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A fully discrete numerical scheme for weighted mean curvature flow

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Summary. We analyze a fully discrete numerical scheme approximating the evolution of n–dimensional graphs under anisotropic mean curvature. The highly nonlinear problem is discretized by piecewise linear finite elements in space and semi–implicitly in time. The scheme is unconditionally stable und we obtain optimal error estimates in natural norms. We also present numerical examples which confirm our theoretical results.

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1. Introduction

The purpose of this paper is to analyze a fully discrete finite element method for the approximation of hypersurfaces $\Gamma_t \subset \mathbb{R}^{n+1}$ which evolve according to the weighted mean curvature flow

(1.1) $$\beta(\nu)V = -H_\gamma \qquad \text{on } \Gamma_t.$$

Here, ν denotes the unit normal to Γ_t and V is the normal velocity of Γ_t. The function $\beta : S^n \to \mathbb{R}$ is positive and continuous. Furthermore, H_γ is the anisotropic mean curvature with respect to the positive, convex and 1–homogeneous weight function $\gamma : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$. We can introduce H_γ formally as the first variation of the weighted area

$$A_\gamma(\Gamma) = \int_\Gamma \gamma(\nu)do.$$

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Associated with the anisotropic surface energy A_γ are the Frank diagram $\mathcal{F}$ and the Wulff shape $\mathcal{W}$ which are given by

$$\mathcal{F} = \{p \in \mathbb{R}^{n+1} \mid \gamma(p) \leq 1\},$$
$$\mathcal{W} = \{p \in \mathbb{R}^{n+1} \mid \langle p, q\rangle \leq \gamma(q) \quad \forall q \in \mathbb{R}^{n+1}\}.$$

It is well–known that the weighted curvature H_γ is constant on $\partial\mathcal{W}$. In Fig. 1 we show Frank diagram and Wulff shape for the anisotropy

$$\gamma(p) = \sqrt{(5.5 + 4.5\text{sign}(p_1))p_1^2 + p_2^2 + p_3^2}. \tag{1.2}$$

Note that for the isotropic case, $\gamma(p) = |p|, \beta \equiv 1$, H_γ becomes the usual mean curvature and (1.1) is the classical mean curvature flow.

A slightly more general law of the form

$$\beta(\nu)V = -H_\gamma + c \tag{1.3}$$

arises for example in the mathematical modelling of the evolution of an interface Γ_t separating a liquid and a solid phase under the assumption that the free energy in either phase is constant. The energy difference in the bulk phases then is given by the constant c, while γ represents the interfacial energy. Finally, the function β measures the drag opposing interfacial motion. For a detailed derivation of (1.3) from the force balances and the second law of thermodynamics see [1]. Let us remark that it is possible to extend the methods presented in this paper to the more general law (1.3), even though we shall not pursue this.

In what follows we shall study surfaces Γ_t which can be described as the graph of a height function $u(\cdot, t)$ over some base domain $\Omega \subset \mathbb{R}^n$, i.e. $\Gamma_t = \{(x, u(x,t)) \,|\, x \in \Omega\}$. The area element and a unit normal are then given by

$$Q(u) = \sqrt{1 + |\nabla u|^2}, \quad \nu(u) = \frac{(\nabla u, -1)}{\sqrt{1 + |\nabla u|^2}} = \frac{(\nabla u, -1)}{Q(u)} \tag{1.4}$$

so that we can calculate the weighted area for a graph Γ given by the height function u as

$$A_\gamma(\Gamma) = A_\gamma(u) = \int_\Omega \gamma(\nu(u))Q(u) = \int_\Omega \gamma(\nabla u, -1)$$

in view of the homogeneity of γ. The first variation of A_γ in the direction of a function $\phi \in C_0^\infty(\Omega)$ then is

$$\frac{d}{d\epsilon} A_\gamma(u + \epsilon\phi)_{|\epsilon=0} = \sum_{i=1}^n \int_\Omega \gamma_{p_i}(\nabla u, -1)\phi_{x_i}$$
$$= -\sum_{i,j=1}^n \int_\Omega \gamma_{p_i p_j}(\nabla u, -1)u_{x_i x_j}\phi = -\int_\Omega H_\gamma \phi. \tag{1.5}$$

In order to translate (1.1) into a differential equation for $u = u(x,t)$ we observe that the normal velocity V of Γ_t is given by

$$V = -\frac{u_t}{Q(u)}. \tag{1.6}$$

Combining (1.5) and (1.6) we arrive at the following initial boundary value problem:

$$\beta(\nu(u))u_t - Q(u)\sum_{i,j=1}^{n} \gamma_{p_i p_j}(\nabla u, -1)u_{x_i x_j} = 0 \quad \text{in } \Omega \times (0,T) \tag{1.7}$$

$$u = u_0 \quad \text{on } \partial\Omega \times (0,T) \tag{1.8}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega. \tag{1.9}$$

Here, $u_0 : \Omega \to \mathbb{R}$ is a given function, so that (1.8) says that the boundary of Γ_t stays fixed during the evolution.

Let us now outline the main ideas of our numerical method. Although the equation (1.7) is not in divergence form it admits the following variational formulation:

$$\int_\Omega \frac{\beta(\nu(u))}{Q(u)} u_t \varphi + \sum_{i=1}^{n} \int_\Omega \gamma_{p_i}(\nu(u))\varphi_{x_i} = 0 \qquad \forall \varphi \in H_0^1(\Omega). \tag{1.10}$$

The variational structure allows the use of finite elements for discretization in space. In order to discretize in time we introduce a semi–implicit scheme which treats the nonlinear terms explicitly. This scheme takes the form

$$\begin{aligned} &\frac{1}{\tau}\int_{\Omega_h} \frac{\beta(\nu(u_h^m))}{Q(u_h^m)}(u_h^{m+1} - u_h^m)\varphi_h + \sum_{i=1}^{n}\int_{\Omega_h} \gamma_{p_i}(\nu(u_h^m))\varphi_{h,x_i} \\ &+\lambda \int_{\Omega_h} \frac{\gamma(\nu(u_h^m))}{Q(u_h^m)} \nabla(u_h^{m+1} - u_h^m)\cdot\nabla\varphi_h = 0 \quad \text{for all } \varphi_h \in \mathring{X}_h. \end{aligned} \tag{1.11}$$

Here, $\tau > 0$ is the time step size and u_h^m belongs to the space of linear finite elements on the discrete domain Ω_h. The scheme (1.11) requires the solution of a linear system in each time step. We shall see in Sect. 3 that it is unconditionally stable provided the parameter λ satisfies the condition

$$\lambda \inf_{|p|=1} \gamma(p) > \bar{\gamma} := \frac{1}{\sqrt{5}-1} \max\{\sup_{|p|=1} |\gamma'(p)|, \sup_{|p|=1} |\gamma''(p)|\}. \tag{1.12}$$

Assuming (1.12) we shall then obtain in Sect. 4 our main result, that under suitable regularity assumptions on the continous solution the following error estimate for the geometric quantities V and ν holds:

$$\sum_{m=0}^{[\frac{T}{\tau}]-1} \tau \int_{\Gamma_h^m} |V^m - V_h^m|^2 do$$
$$+ \max_{0 \le m \le [\frac{T}{\tau}]} \int_{\Gamma_h^m} |\nu(u(\cdot, m\tau)) - \nu(u_h^m)|^2 do \le c(\tau^2 + h^2). \tag{1.13}$$

Here, $\Gamma_h^m = \{(x, u_h^m(x)) \,|\, x \in \Omega \cap \Omega_h\}$ and

$$V^m = -\frac{u_t(\cdot, m\tau)}{Q(u(\cdot, m\tau))}, \quad V_h^m = -\frac{(u_h^{m+1} - u_h^m)/\tau}{Q(u_h^m)}$$

are the continuous and discrete normal velocities respectively.

An error analysis for a semi–discretization in space of (1.7)–(1.9) with $\beta \equiv 1$ is carried out in [4], for the isotropic case see also [3]. An error bound for a fully discrete scheme in the isotropic case is proved in [5] for two–dimensional graphs. It requires a restriction on the time step size of the form $\tau \le \delta h$ as well as inverse estimates. The error bound (1.13), which we are going to prove for the general anisotropic case neither needs any restriction on the dimension of Γ_t nor does it require any relation between τ and the grid size h. Furthermore, only local nondegeneracy of the grid is needed, so that the method is accessible to modern adaptive techniques. The main reason for these improvements is that we are able to close our estimates without having to require $|\nabla u_h^m|$ to be uniformly bounded. This is achieved by a careful use of the weight Q_h^m throughout the analysis.

Let us next refer to other work which is related to the subject of this paper. A number of results have been obtained for the anisotropic evolution of one–dimensional graphs. Giga [9] studies this motion in the case of a nonconvex anisotropy function γ. For such a problem, numerical simulations are carried out in [8]. Analysis and numerical results for a crystalline anisotropy can be found in [7]. In [10] the surface energy γ is approximated by a crystalline one and a convergence analysis for the resulting scheme is given.

Let us finally mention that [2] studies anisotropic motion by mean curvature in the context of Finsler geometry and [15] gives a survey of various mathematical approaches to (1.1).

2. Assumptions

Let us next formulate our assumptions on the data of the problem: we shall assume that $\gamma \in C^3(\mathbb{R}^{n+1} \setminus \{0\})$, $\gamma(p) > 0$ for $p \in \mathbb{R}^{n+1} \setminus \{0\}$ and that γ

is positively homogeneous of degree one, i.e.

$$\gamma(\lambda p) = |\lambda|\gamma(p) \qquad \text{for all } \lambda \neq 0, p \neq 0. \tag{2.1}$$

Here, $|\cdot|$ denotes the Euclidean norm. We shall reserve $\langle \cdot, \cdot \rangle$ to denote the usual scalar product in $\mathbb{R}^{n+1}$. It is not difficult to verify that (2.1) implies

$$\langle \gamma'(p), p \rangle = \gamma(p), \;\; \langle \gamma''(p)p, q \rangle = 0, \tag{2.2}$$

$$\gamma_{p_i}(\lambda p) = \frac{\lambda}{|\lambda|}\gamma_{p_i}(p), \;\; \gamma_{p_ip_j}(\lambda p) = \frac{1}{|\lambda|}\gamma_{p_ip_j}(p) \tag{2.3}$$

for all $p \in \mathbb{R}^{n+1} \setminus \{0\}$, $q \in \mathbb{R}^{n+1}$, $\lambda \neq 0$ and $i, j \in \{1, ..., n+1\}$. Finally, we assume that there exists $\gamma_0 > 0$ such that

$$\langle D^2\gamma(p)q, q \rangle \geq \gamma_0 |q|^2 \qquad \text{for all } p, q \in \mathbb{R}^{n+1}, |p| = 1, \langle p, q \rangle = 0. \tag{2.4}$$

This condition ensures that (1.7) is strictly parabolic, however it is not uniformly parabolic.

Our error analysis will be carried out under the assumption that (1.7)–(1.9) has a solution u which satisfies

$$\begin{aligned} &\sup_{t\in(0,T)} \Big(\|u(\cdot,t)\|_{H^{2,\infty}(\Omega)} + \|u_t(\cdot,t)\|_{H^{1,\infty}(\Omega)} \Big) \\ &\quad + \int_0^T (\|u_t\|^2_{H^2(\Omega)} + \|u_{tt}\|^2) ds =: M < \infty. \end{aligned} \tag{2.5}$$

Here, $H^{m,p}(\Omega)$ denotes the usual Sobolev space. The corresponding norm is given by

$$\|u\|_{H^{m,p}(\Omega)} = \Big(\sum_{k=0}^{m} \|D^k u\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}}$$

with the usual modification for $p = \infty$. For $p = 2$ we simply write $H^m(\Omega) = H^{m,2}(\Omega)$ with norm $\|\cdot\|_{H^m(\Omega)}$; furthermore we use $\|\cdot\|$ to denote the L^2-norm.

Applying a theory developed by Lieberman in [12], the existence and uniqueness of a solution of (1.7)–(1.9) satisfying (2.5) has been obtained in [4] for the case $\beta \equiv 1$ under suitable conditions on $\partial\Omega$ (see also [11] and [13] for the case of mean curvature flow). Under appropriate assumptions on β it is possible to generalize this result to the equation (1.7). Rather than specifying the precise conditions on β that are needed to guarantee (2.5) we just make two assumptions which will be sufficient to carry out the error analysis, namely

$$0 < c_\beta \leq \beta(p) \leq C_\beta < \infty \tag{2.6}$$

$$|\beta(p) - \beta(q)| \leq L_\beta |p - q| \tag{2.7}$$

for all $p, q \in \mathbb{R}^{n+1}, |p| = |q| = 1$.

We now turn to the discretization of (1.7)–(1.9). Let $\mathcal{T}_h$ be a family of triangulations of Ω with maximum mesh size $h := \max_{S \in \mathcal{T}_h} \operatorname{diam}(S)$. We denote by Ω_h the corresponding discrete domain, i.e.

$$\bar{\Omega}_h = \bigcup_{S \in \mathcal{T}_h} S$$

and assume that all vertices on $\partial\Omega_h$ also lie on $\partial\Omega$. Furthermore we suppose that the triangulation is nondegenerate in the sense that

$$\max_{S \in \mathcal{T}_h} \frac{\operatorname{diam}(S)}{\rho_S} \leq \kappa$$

where the constant $\kappa > 0$ is independent of h and ρ_S denotes the radius of the largest ball which is contained in $\bar{S}$.

The discrete space is defined by

$$X_h := \{v_h \in C^0(\bar{\Omega}_h) \,|\, v_h \text{ is a linear polynomial on each } S \in \mathcal{T}_h\}$$

and $\overset{\circ}{X}_h := X_h \cap H^1_0(\Omega_h)$. There exists an interpolation operator $\Pi_h : H^2(\Omega_h) \to X_h$ mapping $H^2(\Omega_h) \cap H^1_0(\Omega)$ into $\overset{\circ}{X}_h$ such that

$$\|v - \Pi_h v\| + h\|\nabla(v - \Pi_h v)\| \leq ch^2 \|v\|_{H^2(\Omega_h)} \quad \text{for all } v \in H^2(\Omega_h). \tag{2.8}$$

Defining $Q^m_h = Q(u^m_h)$ and $\nu^m_h = \nu(u^m_h)$ we are now in position to give a precise formulation of our numerical scheme:

Algorithm 2.1 *given* $u^m_h \in X_h$*, find* $u^{m+1}_h \in X_h$ *such that* $u^{m+1}_h - \Pi_h u_0 \in \overset{\circ}{X}_h$ *and*

$$\frac{1}{\tau} \int_{\Omega_h} \frac{\beta(\nu^m_h)}{Q^m_h} (u^{m+1}_h - u^m_h)\varphi_h + \sum_{i=1}^{n} \int_{\Omega_h} \gamma_{p_i}(\nu^m_h)\varphi_{h,x_i}$$
$$+ \quad \lambda \int_{\Omega_h} \frac{\gamma(\nu^m_h)}{Q^m_h} \nabla(u^{m+1}_h - u^m_h) \cdot \nabla\varphi_h = 0 \qquad \textit{for all } \varphi_h \in \overset{\circ}{X}_h. \tag{2.9}$$

Here we have also set $u^0_h = \Pi_h u_0$ and think of u_0 as being extended to a neighbourhood of Ω.

3. Stability

In this section we analyze the stability of the scheme (2.9). Our main result is the following

Theorem 3.1 *We have for* $M \geq 1$

$$
\begin{aligned}
&\tau \sum_{m=1}^{M-1} \int_{\Omega_h} \beta(\nu_h^m)\, |V_h^m|^2 Q_h^m \\
&\quad + (\lambda \inf_{|p|=1} \gamma(p) - \bar{\gamma})\tau \sum_{m=1}^{M-1} \int_{\Omega_h} |\frac{\nu_h^{m+1} - \nu_h^m}{\sqrt{\tau}}|^2 Q_h^{m+1} \\
&\quad + \lambda\tau \sum_{m=1}^{M-1} \int_{\Omega_h} \frac{\gamma(\nu_h^m)}{Q_h^m} \Big(\frac{Q_h^{m+1} - Q_h^m}{\sqrt{\tau}}\Big)^2 \\
&\quad + \int_{\Omega_h} \gamma(\nu_h^M) Q_h^M \quad \leq \quad \int_{\Omega_h} \gamma(\nu_h^0) Q_h^0.
\end{aligned}
$$

In particular, if λ *satisfies (1.12), then*

$$
\sup_{m \in \mathbb{N}_0} \int_{\Omega_h} Q_h^m \leq C(u_0, \gamma). \tag{3.1}
$$

Proof. Choose $\varphi_h = u_h^{m+1} - u_h^m \in \mathring{X}_h$ in (1.11) and obtain

$$
\begin{aligned}
&\frac{1}{\tau} \int_{\Omega_h} \frac{\beta(\nu_h^m)}{Q_h^m} |u_h^{m+1} - u_h^m|^2 + \sum_{i=1}^{n} \int_{\Omega_h} \gamma_{p_i}(\nu_h^m)(u_h^{m+1} - u_h^m)_{x_i} \\
&\qquad + \lambda \int_{\Omega_h} \frac{\gamma(\nu_h^m)}{Q_h^m} |\nabla(u_h^{m+1} - u_h^m)|^2 = 0.
\end{aligned} \tag{3.2}
$$

Observing that $(\nabla u_h^m, -1) = Q_h^m \nu_h^m$ we can write with the help of (2.1) and (2.2)

$$
\begin{aligned}
\sum_{i=1}^{n} \gamma_{p_i}(\nu_h^m)(u_h^{m+1} - u_h^m)_{x_i} &= \langle \gamma'(\nu_h^m), Q_h^{m+1}\nu_h^{m+1} - Q_h^m \nu_h^m \rangle \\
&= \gamma(\nu_h^{m+1}) Q_h^{m+1} - \gamma(\nu_h^m) Q_h^m + Q_h^{m+1} \langle \gamma'(\nu_h^m), \nu_h^{m+1} \rangle \\
&\quad - Q_h^m \langle \gamma'(\nu_h^m), \nu_h^m \rangle - \gamma(\nu_h^{m+1}) Q_h^{m+1} + \gamma(\nu_h^m) Q_h^m \\
&= \gamma(\nu_h^{m+1}) Q_h^{m+1} - \gamma(\nu_h^m) Q_h^m + R^m\, Q_h^{m+1},
\end{aligned} \tag{3.3}
$$

where

$$
R^m := \langle \gamma'(\nu_h^m), \nu_h^{m+1} \rangle - \gamma(\nu_h^{m+1}).
$$

We estimate R^m from below. A natural way to proceed is to rewrite R^m with the help of second derivatives of γ evaluated on the segment connecting ν_h^m and ν_h^{m+1}. In view of (2.3) such an expression is critical if the segment is close to the origin. In order to deal with this problem we distinguish between two cases:

Case 1 $|\nu_h^{m+1} - \nu_h^m|^2 < \alpha$ ($0 < \alpha < 4$ to be chosen later). Since $|\nu_h^m| = |\nu_h^{m+1}| = 1$ we observe that for every $s \in [0,1]$

$$|(1-s)\nu_h^m + s\,\nu_h^{m+1}|^2 = 1 - s(1-s)|\nu_h^{m+1} - \nu_h^m|^2 \geq 1 - s(1-s)\alpha \geq 1 - \frac{\alpha}{4}, \tag{3.4}$$

so that using (2.2) and (2.3)

$$\begin{aligned} R^m &= -\Big(\gamma(\nu_h^{m+1}) - \gamma(\nu_h^m) - \langle \gamma'(\nu_h^m), \nu_h^{m+1} - \nu_h^m \rangle\Big) \\ &= -\int_0^1 (1-s)\langle \gamma''((1-s)\nu_h^m + s\nu_h^{m+1})(\nu_h^{m+1} - \nu_h^m), \\ &\qquad (\nu_h^{m+1} - \nu_h^m)\rangle ds \\ &\geq -\sup_{|p|=1} |\gamma''(p)| \int_0^1 \frac{(1-s)}{|(1-s)\nu_h^m + s\nu_h^{m+1}|} ds\, |\nu_h^{m+1} - \nu_h^m|^2 \\ &\geq -\frac{1}{\sqrt{4-\alpha}} \sup_{|p|=1} |\gamma''(p)|\, |\nu_h^{m+1} - \nu_h^m|^2. \end{aligned} \tag{3.5}$$

Case 2 $|\nu_h^{m+1} - \nu_h^m|^2 \geq \alpha$. We deduce from (2.2)

$$\begin{aligned} R^m &= \langle \gamma'(\nu_h^m) - \gamma'(\nu_h^{m+1}), \nu_h^{m+1} \rangle \geq -|\gamma'(\nu_h^m) - \gamma'(\nu_h^{m+1})| \\ &\geq -2 \sup_{|p|=1} |\gamma'(p)| \geq -\frac{2}{\alpha} \sup_{|p|=1} |\gamma'(p)|\, |\nu_h^{m+1} - \nu_h^m|^2. \end{aligned} \tag{3.6}$$

Let us choose $\alpha = 2(\sqrt{5} - 1)$. Then $\frac{2}{\alpha} = \frac{1}{\sqrt{4-\alpha}}$ and we obtain from (3.3), (3.5) and (3.6)

$$\begin{aligned} \sum_{i=1}^n \gamma_{p_i}(\nu_h^m)(u_h^{m+1} - u_h^m)_{x_i} &\geq \gamma(\nu_h^{m+1})Q_h^{m+1} - \gamma(\nu_h^m)Q_h^m \\ &\quad - \bar{\gamma}|\nu_h^{m+1} - \nu_h^m|^2 Q_h^{m+1} \end{aligned}$$

Thus, (3.2) combined with (A.2) yields

$$\begin{aligned} \frac{1}{\tau}\int_{\Omega_h} \frac{\beta(\nu_h^m)}{Q_h^m}|u_h^{m+1} - u_h^m|^2 &+ \int_{\Omega_h} \gamma(\nu_h^{m+1})Q_h^{m+1} - \int_{\Omega_h} \gamma(\nu_h^m)Q_h^m \\ &+ \int_{\Omega_h} (\lambda\gamma(\nu_h^m) - \bar{\gamma})|\nu_h^{m+1} - \nu_h^m|^2\, Q_h^{m+1} \\ &+ \int_{\Omega_h} \lambda\gamma(\nu_h^m)\, \frac{(Q_h^{m+1} - Q_h^m)^2}{Q_h^m} \leq 0, \end{aligned}$$

from which the theorem follows by summing over $m = 0, ..., M-1$. □

Thus we have proved stability for the semi–implicit scheme without any restriction on the time step size.

Example 3.2 In [5] we analyzed the following scheme for the isotropic case $\gamma(p) = |p|$ and $\beta \equiv 1$:

$$\frac{1}{\tau}\int_{\Omega_h} \frac{(u_h^{m+1} - u_h^m)\varphi_h}{Q_h^m} + \int_{\Omega_h} \frac{\nabla u_h^{m+1} \cdot \nabla \varphi_h}{Q_h^m} = 0$$

which is (1.11) for the choice $\lambda = 1$. Since $\inf_{|p|=1} \gamma(p) = 1$, $\sup_{|p|=1} |\gamma'(p)| = \sup_{|p|=1} |\gamma''(p)| = 1$ and $1 > \frac{1}{\sqrt{5}-1}$ we recover the unconditional stability of this scheme (see also [6]).

4. The error estimate

The aim of this section is to carry out an error analysis for the scheme (1.11). In what follows we shall assume that the condition (1.12) is satisfied, so that the scheme (1.11) is stable. Choose on open set $\Omega' \subset \mathbb{R}^n$ which contains $\bar{\Omega} \cup \Omega_h$ for all $h \leq 1$. In view of the regularity (2.5) of u and since $\partial\Omega$ is smooth, there exists an extension $\bar{u} : \Omega' \times [0, T] \to \mathbb{R}$ such that $\bar{u}_{|\Omega\times[0,T]} = u$ and

$$\begin{aligned} &\sup_{t\in(0,T)} \Big(\|\bar{u}(\cdot, t)\|_{H^{2,\infty}(\Omega')} + \|\bar{u}_t(\cdot, t)\|_{H^{1,\infty}(\Omega')} \Big) \\ &\quad + \int_0^{\mathrm{T}} (\|\bar{u}_t\|^2_{H^2(\Omega')} + \|\bar{u}_{tt}\|^2) ds \leq cM, \end{aligned} \tag{4.1}$$

where M appeared in (2.5). Using integration by parts and (1.7) we derive for $\varphi \in H^1_0(\Omega_h)$

$$\begin{aligned} \sum_{i=1}^n \int_{\Omega_h} \gamma_{p_i}(\bar{\nu}^m)\varphi_{x_i} &= -\sum_{i=1}^n \int_{\Omega_h} \frac{\partial}{\partial x_i}(\gamma_{p_i}(\bar{\nu}^m))\varphi \\ &= -\int_{\Omega_h\cap\Omega} \frac{\beta(\nu^m)}{Q^m} u_t(\cdot, m\tau)\varphi \\ &\quad -\sum_{i=1}^n \int_{\Omega_h\setminus\Omega} \frac{\partial}{\partial x_i}(\gamma_{p_i}(\bar{\nu}^m))\varphi \\ &= -\frac{1}{\tau}\int_{\Omega_h} \bar{\alpha}^m(\bar{u}^{m+1} - \bar{u}^m)\varphi + \int_{\Omega_h} \bar{\alpha}^m S^m \varphi \end{aligned} \tag{4.2}$$

where we have set $\bar{\alpha}^m = \frac{\beta(\bar{\nu}^m)}{\bar{Q}^m}$ and

$$
\begin{aligned}
S^m &= \frac{\bar{u}^{m+1} - \bar{u}^m}{\tau} - \bar{u}_t(\cdot, m\tau) \\
&\quad + \chi_{\Omega_h \setminus \Omega}\Big(\bar{u}_t(\cdot, m\tau) - \frac{1}{\bar{\alpha}^m}\sum_{i=1}^n \frac{\partial}{\partial x_i}(\gamma_{p_i}(\bar{\nu}^m))\Big) \qquad (4.3)\\
&= \frac{1}{\tau}\int_{m\tau}^{(m+1)\tau} ((m+1)\tau - s)\bar{u}_{tt}(\cdot, s)ds \\
&\quad + \chi_{\Omega_h \setminus \Omega}\Big(\bar{u}_t(\cdot, m\tau) - \frac{1}{\bar{\alpha}^m}\sum_{i=1}^n \frac{\partial}{\partial x_i}(\gamma_{p_i}(\bar{\nu}^m))\Big).
\end{aligned}
$$

Note that (4.3) and (4.1) imply

$$
\begin{aligned}
\|S^m\|_{L^\infty(\Omega_h)} &\le c(\|\bar{u}_t(\cdot, m\tau)\|_{L^\infty(\Omega_h)} \\
&\quad + \|D^2\bar{u}(\cdot, m\tau)\|_{L^\infty(\Omega_h)}) \le cM \qquad (4.4)
\end{aligned}
$$

$$
\|S^m\| \le c\Big(\tau \int_{m\tau}^{(m+1)\tau} \|\bar{u}_{tt}\|^2 ds\Big)^{\frac{1}{2}} + ch \qquad (4.5)
$$

since meas$(\Omega_h \setminus \Omega) \le ch^2$. In order to derive the error relation we write

$$
\mathrm{e}^m := \bar{u}^m - u_h^m = (\bar{u}^m - \Pi_h \bar{u}^m) + (\Pi_h \bar{u}^m - u_h^m) =: \epsilon^m + \mathrm{e}_h^m. \qquad (4.6)
$$

Note that (2.8) implies

$$
\begin{aligned}
&\|\varepsilon^{m+1} - \varepsilon^m\|^2 + h^2\|\nabla(\varepsilon^{m+1} - \varepsilon^m)\|^2 \\
&\quad \le c\tau h^4 \int_{m\tau}^{(m+1)\tau} \|\bar{u}_t\|^2_{H^2(\Omega_h)} ds. \qquad (4.7)
\end{aligned}
$$

Combining (4.2) and (1.11) we arrive at the following error relation

$$
\begin{aligned}
&\frac{1}{\tau}\int_{\Omega_h} \alpha_h^m(\mathrm{e}^{m+1} - \mathrm{e}^m)\varphi_h + \sum_{i=1}^n \int_{\Omega_h} (\gamma_{p_i}(\bar{\nu}^m) - \gamma_{p_i}(\nu_h^m))\varphi_{h,x_i} \\
&\quad + \lambda \int_{\Omega_h} \frac{\gamma(\nu_h^m)}{Q_h^m}\nabla(\mathrm{e}^{m+1} - \mathrm{e}^m)\cdot\nabla\varphi_h \\
&= \frac{1}{\tau}\int_{\Omega_h} (\alpha_h^m - \bar{\alpha}^m)(\bar{u}^{m+1} - \bar{u}^m)\varphi_h \\
&\quad + \lambda \int_{\Omega_h} \frac{\gamma(\nu_h^m)}{Q_h^m}\nabla(\bar{u}^{m+1} - \bar{u}^m)\cdot\nabla\varphi_h \\
&\quad + \int_{\Omega_h} \bar{\alpha}^m S^m \varphi_h
\end{aligned}
$$

for all $\varphi_h \in \mathring{X}_h$. Here, we have also set $\alpha_h^m = \frac{\beta(\nu_h^m)}{Q_h^m}$. If we insert $\varphi_h = e_h^{m+1} - e_h^m \in \mathring{X}_h$ the result is

$$\begin{aligned}
&\frac{1}{\tau}\int_{\Omega_h} \alpha_h^m |\mathrm{e}^{m+1} - \mathrm{e}^m|^2 + \sum_{i=1}^n \int_{\Omega_h} (\gamma_{p_i}(\bar{\nu}^m) - \gamma_{p_i}(\nu_h^m))(\mathrm{e}^{m+1} - \mathrm{e}^m)_{x_i} \\
&\quad + \lambda \int_{\Omega_h} \frac{\gamma(\nu_h^m)}{Q_h^m} |\nabla(\mathrm{e}^{m+1} - \mathrm{e}^m)|^2 \\
&= \frac{1}{\tau}\int_{\Omega_h} (\alpha_h^m - \bar{\alpha}^m)(\bar{u}^{m+1} - \bar{u}^m)(e_h^{m+1} - e_h^m) \\
&\quad + \lambda \int_{\Omega_h} \frac{\gamma(\nu_h^m)}{Q_h^m} \nabla(\bar{u}^{m+1} - \bar{u}^m) \cdot \nabla(e_h^{m+1} - e_h^m) \\
&\quad + \int_{\Omega_h} \bar{\alpha}^m S^m (e_h^{m+1} - e_h^m) + \frac{1}{\tau}\int_{\Omega_h} \alpha_h^m (\mathrm{e}^{m+1} - \mathrm{e}^m)(\epsilon^{m+1} - \epsilon^m) \\
&\quad + \sum_{i=1}^n \int_{\Omega_h} (\gamma_{p_i}(\bar{\nu}^m) - \gamma_{p_i}(\nu_h^m))(\varepsilon^{m+1} - \varepsilon^m)_{x_i} \\
&\quad + \lambda \int_{\Omega_h} \frac{\gamma(\nu_h^m)}{Q_h^m} \nabla(\mathrm{e}^{m+1} - \mathrm{e}^m) \cdot \nabla(\varepsilon^{m+1} - \varepsilon^m).
\end{aligned} \tag{4.8}$$

Our aim is to write the second term on the left hand side of (4.8) as a discrete time derivative plus a remainder. Since the expression $\gamma_{p_i}(\nu(u))$ depends in a nonlinear way on ∇u this is by no means straightforward. The idea is to consider

$$D^m := \int_{\Omega_h} (\gamma(\nu_h^m) - \langle \gamma'(\bar{\nu}^m), \nu_h^m \rangle) Q_h^m$$

which is motivated by the following two results:

Lemma 4.1 *There exists a constant* $c_1 > 0$ *which only depends on* $\sup_{\Omega' \times [0,T]} |\nabla \bar{u}|$ *and* γ_0 *from (2.4) such that*

$$D^m \geq c_1 \int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|^2 Q_h^m$$

Proof. See Lemma 3.2 in [4]. □

Lemma 4.2 *We have for* $0 \leq m \leq [\frac{T}{\tau}] - 1$ *and small* $\tau > 0$

$$\sum_{i=1}^n \int_{\Omega_h} (\gamma_{p_i}(\bar{\nu}^m) - \gamma_{p_i}(\nu_h^m))(\mathrm{e}^{m+1} - \mathrm{e}^m)_{x_i} \quad \geq \quad D^{m+1} - D^m$$

$$-c\tau\Big(\tau^2 + \int_{\Omega_h} |\,\bar\nu^{m+1} - \nu_h^{m+1}|^2 Q_h^{m+1}\Big)$$
$$-(\bar\gamma + c\tau)\int_{\Omega_h} \frac{|\nabla(\mathrm{e}^{m+1} - \mathrm{e}^m)|^2}{Q_h^m}.$$

Proof. Let us abbreviate $\bar U^m = (\nabla \bar u^m, -1),\ U_h^m = (\nabla u_h^m, -1)$. Clearly,

$$\sum_{i=1}^n (\gamma_{p_i}(\bar\nu^m) - \gamma_{p_i}(\nu_h^m))(\mathrm{e}^{m+1} - \mathrm{e}^m)_{x_i}$$
$$= \langle \gamma'(\bar\nu^m) - \gamma'(\nu_h^m), (\bar U^{m+1} - U_h^{m+1}) - (\bar U^m - U_h^m)\rangle$$
$$= (\gamma(U_h^{m+1}) - \langle\gamma'(\bar\nu^{m+1}), U_h^{m+1}\rangle) - (\gamma(U_h^m) - \langle\gamma'(\bar\nu^m), U_h^m\rangle)$$
$$+ \gamma(U_h^m) - \gamma(U_h^{m+1}) + \langle\gamma'(\nu_h^m), U_h^{m+1} - U_h^m\rangle$$
$$+\langle\gamma'(\bar\nu^{m+1}) - \gamma'(\bar\nu^m), U_h^{m+1}\rangle$$
$$+\langle\gamma'(\bar\nu^m) - \gamma'(\nu_h^m), \bar U^{m+1} - \bar U^m\rangle$$
$$= (\gamma(\nu_h^{m+1}) - \langle\gamma'(\bar\nu^{m+1}), \nu_h^{m+1}\rangle)Q_h^{m+1}$$
$$-(\gamma(\nu_h^m) - \langle\gamma'(\bar\nu^m), \nu_h^m\rangle)Q_h^m + I^m, \tag{4.9}$$

by (2.1) and since $U_h^m = \nu_h^m Q_h^m$. Here we have also set

$$I^m = \gamma(U_h^m) - \gamma(U_h^{m+1}) + \langle\gamma'(\nu_h^m), U_h^{m+1} - U_h^m\rangle$$
$$+\langle\gamma'(\bar\nu^{m+1}) - \gamma'(\bar\nu^m), U_h^{m+1}\rangle + \langle\gamma'(\bar\nu^m) - \gamma'(\nu_h^m), \bar U^{m+1} - \bar U^m\rangle.$$

The first two terms in (4.9) will give the difference $D^{m+1} - D^m$ after integration over Ω_h. In order to estimate I^m we proceed in the same way as in the proof of Theorem 3.1 and distinguish between two cases:
Case 1 $|\nu_h^{m+1} - \nu_h^m|^2 < \alpha = 2(\sqrt{5} - 1)$: first recall that (3.4) yields

$$|s\nu_h^{m+1} + (1-s)\nu_h^m| \ge \sqrt{1 - \frac{\alpha}{4}} > 0 \quad \text{for all } s \in [0,1]. \tag{4.10}$$

We rewrite the three terms which compose I^m. Since $U_h^m = \nu_h^m Q_h^m, \bar U^m = \bar\nu^m \bar Q^m$ we derive with the help of (2.2)

$$\gamma(U_h^m) - \gamma(U_h^{m+1}) + \langle\gamma'(\nu_h^m), U_h^{m+1} - U_h^m\rangle$$
$$= -Q_h^{m+1}\Big(\gamma(\nu_h^{m+1}) - \langle\gamma'(\nu_h^m), \nu_h^{m+1}\rangle\Big)$$
$$= -Q_h^{m+1}\Big(\gamma(\nu_h^{m+1}) - \gamma(\nu_h^m) - \langle\gamma'(\nu_h^m), \nu_h^{m+1} - \nu_h^m\rangle\Big)$$
$$= -Q_h^{m+1}\int_0^1 (1-s)\langle\gamma''(s\nu_h^{m+1} + (1-s)\nu_h^m)(\nu_h^{m+1} - \nu_h^m),$$
$$(\nu_h^{m+1} - \nu_h^m)\rangle ds.$$

Furthermore, observing that $\langle\gamma''(p)p,q\rangle = 0$ for all $p,q \in \mathbb{R}^{n+1}, p \neq 0$ we obtain

$$\begin{aligned}
&\langle\gamma'(\bar\nu^{m+1}) - \gamma'(\bar\nu^m), U_h^{m+1}\rangle \\
&= Q_h^{m+1}\int_0^1 \langle\gamma''(s\bar\nu^{m+1} + (1-s)\bar\nu^m)(\bar\nu^{m+1} - \bar\nu^m), \nu_h^{m+1}\rangle ds \\
&= Q_h^{m+1}\int_0^1 s\langle\gamma''(s\bar\nu^{m+1} + (1-s)\bar\nu^m)(\bar\nu^{m+1} - \bar\nu^m), \\
&\qquad (\nu_h^{m+1} - \bar\nu^{m+1})\rangle ds \\
&\quad + Q_h^{m+1}\int_0^1 (1-s)\langle\gamma''(s\bar\nu^{m+1} + (1-s)\bar\nu^m)(\bar\nu^{m+1} - \bar\nu^m), \\
&\qquad (\nu_h^{m+1} - \bar\nu^m)\rangle ds.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\langle\gamma'(\bar\nu^m) - \gamma'(\nu_h^m), \bar U^{m+1} - \bar U^m\rangle \\
&= \bar Q^{m+1}\langle\gamma'(\bar\nu^m) - \gamma'(\nu_h^m), \bar\nu^{m+1} - \bar\nu^m\rangle \\
&\quad +(\bar Q^{m+1} - \bar Q^m)\langle\gamma'(\bar\nu^m) - \gamma'(\nu_h^m), \bar\nu^m\rangle \\
&= Q_h^{m+1}\int_0^1 \langle\gamma''(s\bar\nu^m + (1-s)\nu_h^m)(\bar\nu^m - \nu_h^m), (\bar\nu^{m+1} - \bar\nu^m)\rangle ds \\
&\quad + (\bar Q^{m+1} - Q_h^{m+1})\int_0^1 \langle\gamma''(s\bar\nu^m + (1-s)\nu_h^m)(\bar\nu^m - \nu_h^m), \\
&\qquad (\bar\nu^{m+1} - \bar\nu^m)\rangle ds \\
&\quad + (\bar Q^{m+1} - \bar Q^m)\int_0^1 (1-s)\langle\gamma''(s\bar\nu^m + (1-s)\nu_h^m)(\bar\nu^m - \nu_h^m), \\
&\qquad (\bar\nu^m - \nu_h^m)\rangle ds
\end{aligned}$$

where we have again used (2.2) in order to show that

$$\begin{aligned}
\langle\gamma'(\bar\nu^m) - \gamma'(\nu_h^m), \bar\nu^m\rangle &= \gamma(\bar\nu^m) - \gamma(\nu_h^m) - \langle\gamma'(\nu_h^m), \bar\nu^m - \nu_h^m\rangle \\
&= \int_0^1 (1-s)\langle\gamma''(s\bar\nu^m + (1-s)\nu_h^m)(\bar\nu^m - \nu_h^m), \\
&\qquad (\bar\nu^m - \nu_h^m)\rangle ds.
\end{aligned}$$

Abbreviating

$$A := \int_0^1 (1-s)\gamma''(s\nu_h^{m+1} + (1-s)\nu_h^m) ds$$

we can write

$$\begin{aligned}
I^m = Q_h^{m+1}\Big(&-\langle A(\nu_h^{m+1}-\nu_h^m),(\nu_h^{m+1}-\nu_h^m)\rangle \\
&+\langle A(\bar\nu^{m+1}-\bar\nu^m),(\nu_h^{m+1}-\bar\nu^{m+1})\rangle \\
&+\langle A(\bar\nu^{m+1}-\bar\nu^m),(\nu_h^{m+1}-\bar\nu^m)\rangle \\
&+2\langle A(\bar\nu^m-\nu_h^m),(\bar\nu^{m+1}-\bar\nu^m)\rangle\Big) + \sum_{i=1}^5 R_i \\
= -Q_h^{m+1}&\langle A((\bar\nu^{m+1}-\bar\nu^m)-(\nu_h^{m+1}-\nu_h^m)), \\
&((\bar\nu^{m+1}-\bar\nu^m)-(\nu_h^{m+1}-\nu_h^m))\rangle + \sum_{i=1}^5 R_i,
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= Q_h^{m+1}\langle\Big(\int_0^1 s\gamma''(s\bar\nu^{m+1}+(1-s)\bar\nu^m)ds - A\Big)(\bar\nu^{m+1}-\bar\nu^m), \\
&\quad (\nu_h^{m+1}-\bar\nu^{m+1})\rangle \\
R_2 &= Q_h^{m+1}\langle\Big(\int_0^1 (1-s)\gamma''(s\bar\nu^{m+1}+(1-s)\bar\nu^m)ds - A\Big)(\bar\nu^{m+1}-\bar\nu^m), \\
&\quad (\nu_h^{m+1}-\bar\nu^m)\rangle \\
R_3 &= Q_h^{m+1}\langle\Big(\int_0^1 \gamma''(s\bar\nu^m+(1-s)\nu_h^m)ds - 2A\Big)(\bar\nu^m-\nu_h^m), \\
&\quad (\bar\nu^{m+1}-\bar\nu^m)\rangle \\
R_4 &= (\bar Q^{m+1}-Q_h^{m+1})\int_0^1 \langle\gamma''(s\bar\nu^m+(1-s)\nu_h^m)(\bar\nu^m-\nu_h^m), \\
&\quad (\bar\nu^{m+1}-\bar\nu^m)\rangle ds \\
R_5 &= (\bar Q^{m+1}-\bar Q^m)\int_0^1 (1-s)\langle\gamma''(s\bar\nu^m+(1-s)\nu_h^m)(\bar\nu^m-\nu_h^m), \\
&\quad (\bar\nu^m-\nu_h^m)\rangle ds.
\end{aligned}$$

Let us estimate $R_i (i=1,\dots,5)$. To begin, note that Lemma 6.1 and (4.10) imply

$$\begin{aligned}
&\Big|\int_0^1 s\gamma''(s\bar\nu^{m+1}+(1-s)\bar\nu^m)ds - A\Big| \\
&\le \Big|\int_0^1 s(\gamma''(s\bar\nu^{m+1}+(1-s)\bar\nu^m)-\gamma''(\bar\nu^{m+1}))ds\Big| \\
&\quad +\frac{1}{2}|\gamma''(\bar\nu^{m+1})-\gamma''(\nu_h^{m+1})|
\end{aligned}$$

$$
\begin{aligned}
&+\Big|\int_0^1 (1-s)(\gamma''(\nu_h^{m+1})-\gamma''(s\nu_h^{m+1}+(1-s)\nu_h^m))ds\Big| \\
&\le c\sup_{|p|=1}|\gamma'''(p)|\Big(|\bar\nu^{m+1}-\bar\nu^m|+|\bar\nu^{m+1}-\nu_h^{m+1}|+|\nu_h^{m+1}-\nu_h^m|\Big) \\
&\le c\Big(\tau+|\bar\nu^{m+1}-\nu_h^{m+1}|+|(\bar\nu^{m+1}-\bar\nu^m)-(\nu_h^{m+1}-\nu_h^m)|\Big),
\end{aligned}
$$

so that

$$
\begin{aligned}
|R_1| \le\ & c\tau Q_h^{m+1}|\nu_h^{m+1}-\bar\nu^{m+1}| \\
&\times\Big(\tau+|\bar\nu^{m+1}-\nu_h^{m+1}|+|(\bar\nu^{m+1}-\bar\nu^m)-(\nu_h^{m+1}-\nu_h^m)|\Big) \\
&\le c\tau Q_h^{m+1} \\
&\times\Big(\tau^2+|\bar\nu^{m+1}-\nu_h^{m+1}|^2+|(\bar\nu^{m+1}-\bar\nu^m)-(\nu_h^{m+1}-\nu_h^m)|^2\Big).
\end{aligned}
$$

Since $|\nu_h^{m+1}-\bar\nu^m|\le|\nu_h^{m+1}-\bar\nu^{m+1}|+c\tau$ we obtain in the same way

$$
|R_2|\le c\tau Q_h^{m+1}\Big(\tau^2+|\bar\nu^{m+1}-\nu_h^{m+1}|^2+|(\bar\nu^{m+1}-\bar\nu^m)-(\nu_h^{m+1}-\nu_h^m)|^2\Big).
$$

For the next term,

$$
\begin{aligned}
&\Big|\int_0^1\gamma''(s\bar\nu^m+(1-s)\nu_h^m)ds-2A\Big| \\
&\le\Big|\int_0^1(\gamma''(s\bar\nu^m+(1-s)\nu_h^m)-\gamma''(\bar\nu^m))ds\Big| \\
&+|\gamma''(\bar\nu^m)-\gamma''(\nu_h^m)| \\
&\quad+\Big|\int_0^1 2(1-s)(\gamma''(\nu_h^m)-\gamma''(s\nu_h^{m+1}+(1-s)\nu_h^m))ds\Big| \\
&\le c\sup_{|p|=1}|\gamma'''(p)|\Big(|\bar\nu^m-\nu_h^m|+|\nu_h^{m+1}-\nu_h^m|\Big) \\
&\le c\Big(|\bar\nu^m-\nu_h^m|+\tau+|(\bar\nu^{m+1}-\bar\nu^m)-(\nu_h^{m+1}-\nu_h^m)|\Big)
\end{aligned}
$$

again by Lemma 6.1 and (4.10). Since

$$
\begin{aligned}
|\bar\nu^m-\nu_h^m|\le\ & c\tau+|\bar\nu^{m+1}-\nu_h^{m+1}| \\
&+|(\bar\nu^{m+1}-\bar\nu^m)-(\nu_h^{m+1}-\nu_h^m)|,
\end{aligned}
\tag{4.11}
$$

we obtain

$$
|R_3|\le c\tau Q_h^{m+1}\Big(\tau^2+|\bar\nu^{m+1}-\nu_h^{m+1}|^2+|(\bar\nu^{m+1}-\bar\nu^m)-(\nu_h^{m+1}-\nu_h^m)|^2\Big).
$$

Using Lemma 6.1, (A.1) and (4.11) we derive

$$\begin{aligned}
|R_4| &\le c|\bar{Q}^{m+1} - Q_h^{m+1}|\,|\bar{\nu}^m - \nu_h^m|\,|\bar{\nu}^{m+1} - \bar{\nu}^m| \\
&\le c\tau Q_h^{m+1}|\bar{\nu}^{m+1} - \nu_h^{m+1}|\,|\bar{\nu}^m - \nu_h^m| \\
&\le c\tau Q_h^{m+1}|\bar{\nu}^{m+1} - \nu_h^{m+1}| \\
&\times \Big(\tau + |\bar{\nu}^{m+1} - \nu_h^{m+1}| + |(\bar{\nu}^{m+1} - \bar{\nu}^m) - (\nu_h^{m+1} - \nu_h^m)|\Big) \\
&\le c\tau Q_h^{m+1} \\
&\times \Big(\tau^2 + |\bar{\nu}^{m+1} - \nu_h^{m+1}|^2 + |(\bar{\nu}^{m+1} - \bar{\nu}^m) - (\nu_h^{m+1} - \nu_h^m)|^2\Big).
\end{aligned}$$

Finally, again by (4.11)

$$\begin{aligned}
&|R_5| \le c|\bar{Q}^{m+1} - \bar{Q}^m|\,|\bar{\nu}^m - \nu_h^m|^2 \quad \le \quad c\tau|\bar{\nu}^m - \nu_h^m|^2 \\
&\le c\tau\Big(\tau^2 + |\bar{\nu}^{m+1} - \nu_h^{m+1}|^2 + |(\bar{\nu}^{m+1} - \bar{\nu}^m) - (\nu_h^{m+1} - \nu_h^m)|^2\Big) \\
&\le c\tau Q_h^{m+1}\Big(\tau^2 + |\bar{\nu}^{m+1} - \nu_h^{m+1}|^2 + |(\bar{\nu}^{m+1} - \bar{\nu}^m) - (\nu_h^{m+1} - \nu_h^m)|^2\Big).
\end{aligned}$$

Collecting the above estimates for $R_1, \ldots, R_5$ and observing that A is positive semidefinite we arrive at

$$\begin{aligned}
|I^m| &\le Q_h^{m+1}\langle A((\bar{\nu}^{m+1} - \bar{\nu}^m) - (\nu_h^{m+1} - \nu_h^m)), \\
&\qquad (\bar{\nu}^{m+1} - \bar{\nu}^m) - (\nu_h^{m+1} - \nu_h^m)\rangle + \sum_{i=1}^{5} |R_i| \\
&\le Q_h^{m+1}\langle A((\bar{\nu}^{m+1} - \bar{\nu}^m) - (\nu_h^{m+1} - \nu_h^m)), \\
&\qquad (\bar{\nu}^{m+1} - \bar{\nu}^m) - (\nu_h^{m+1} - \nu_h^m)\rangle \\
&\qquad + c\tau Q_h^{m+1}\Big(\tau^2 + |\bar{\nu}^{m+1} - \nu_h^{m+1}|^2 \\
&\qquad + |(\bar{\nu}^{m+1} - \bar{\nu}^m) - (\nu_h^{m+1} - \nu_h^m)|^2\Big).
\end{aligned} \tag{4.12}$$

From (2.3) and (4.10) we infer that for every $\xi \in \mathbb{R}^{n+1}$

$$\begin{aligned}
\langle A\xi, \xi\rangle &= \int_0^1 (1-s)\langle \gamma''(s\nu_h^{m+1} + (1-s)\nu_h^m)\xi, \xi\rangle ds \\
&\le \sup_{|p|=1} |\gamma''(p)| \int_0^1 \frac{(1-s)|\xi|^2}{|s\nu_h^{m+1} + (1-s)\nu_h^m|} ds \\
&\le \sup_{|p|=1} |\gamma''(p)| \, \frac{1}{\sqrt{4-\alpha}} \, |\xi|^2.
\end{aligned}$$

If we insert this into (4.12) the result is

$$
\begin{aligned}
|I^m| \le & \Big(\sup_{|p|=1} |\gamma''(p)| \, \frac{1}{\sqrt{4-\alpha}} + c\tau \Big) \\
& \times Q_h^{m+1} |(\bar\nu^{m+1} - \bar\nu^m) - (\nu_h^{m+1} - \nu_h^m)|^2 \\
& + \quad c\tau Q_h^{m+1} \Big(\tau^2 + |\bar\nu^{m+1} - \nu_h^{m+1}|^2 \Big).
\end{aligned} \tag{4.13}
$$

Case 2 $|\nu_h^{m+1} - \nu_h^m|^2 \ge \alpha$: first observe that by (2.2)

$$
\begin{aligned}
& \gamma(U_h^m) - \gamma(U_h^{m+1}) + \langle \gamma'(\nu_h^m), U_h^{m+1} - U_h^m \rangle \\
& = \langle \gamma'(\nu_h^m), U_h^{m+1} \rangle - \gamma(U_h^{m+1}) = \langle \gamma'(\nu_h^m) - \gamma'(\nu_h^{m+1}), \nu_h^{m+1} \rangle Q_h^{m+1},
\end{aligned}
$$

so that

$$
\begin{aligned}
|I^m| \le & |\langle \gamma'(\nu_h^m) - \gamma'(\nu_h^{m+1}), \nu_h^{m+1} \rangle Q_h^{m+1}| + |\langle \gamma'(\bar\nu^{m+1}) \\
& -\gamma'(\bar\nu^m), U_h^{m+1} \rangle| + \, |\langle \gamma'(\bar\nu^m) - \gamma'(\nu_h^m), \bar U^{m+1} - \bar U^m \rangle| \\
\le & \, 2 \sup_{|p|=1} |\gamma'(p)| \, Q_h^{m+1} + c\tau Q_h^{m+1} + c\tau \sup_{|p|=1} |\gamma'(p)| \\
\le & \, (\frac{2}{\alpha} \sup_{|p|=1} |\gamma'(p)| + c\tau) |\nu_h^{m+1} - \nu_h^m|^2 Q_h^{m+1} \\
\le & \, (\frac{2}{\alpha} \sup_{|p|=1} |\gamma'(p)| + c\tau) |(\nu_h^{m+1} - \nu_h^m) - (\bar\nu^{m+1} - \bar\nu^m)|^2 Q_h^{m+1}
\end{aligned} \tag{4.14}
$$

provided τ is small enough.

Combining (4.9), (4.14), (4.13) and applying Lemma 6.4 we finally obtain

$$
\begin{aligned}
& \sum_{i=1}^n (\gamma_{p_i}(\bar\nu^m) - \gamma_{p_i}(\nu_h^m))(\mathrm{e}^{m+1} - \mathrm{e}^m)_{x_i} \\
& \ge (\gamma(\nu_h^{m+1}) - \langle \gamma'(\bar\nu^{m+1}), \nu_h^{m+1} \rangle) Q_h^{m+1} \\
& \quad -(\gamma(\nu_h^m) - \langle \gamma'(\bar\nu^m), \nu_h^m \rangle) Q_h^m - \Big(\bar\gamma + c\tau \Big) Q_h^{m+1} |(\bar\nu^{m+1} - \bar\nu^m) \\
& \quad -(\nu_h^{m+1} - \nu_h^m)|^2 - c\tau Q_h^{m+1} \Big(\tau^2 + |\bar\nu^{m+1} - \nu_h^{m+1}|^2 \Big) \\
& \ge (\gamma(\nu_h^{m+1}) - \langle \gamma'(\bar\nu^{m+1}), \nu_h^{m+1} \rangle) Q_h^{m+1} - (\gamma(\nu_h^m) - \langle \gamma'(\bar\nu^m), \nu_h^m \rangle) Q_h^m \\
& \quad - \Big(\bar\gamma + c\tau \Big) \frac{|\nabla(\mathrm{e}^{m+1} - \mathrm{e}^m)|^2}{Q_h^m} - c\tau Q_h^{m+1} \Big(\tau^2 + |\bar\nu^{m+1} - \nu_h^{m+1}|^2 \Big).
\end{aligned}
$$

The lemma now follows from integrating this inequality over Ω_h and recalling the stability estimate (3.1). □

We estimate the terms on the right hand side of (4.8). To begin, (2.7), (4.6), Young's inequality and (4.7) imply for $0 < \delta \le 1$

$$\begin{aligned}
&\Big|\frac{1}{\tau}\int_{\Omega_h}(\alpha_h^m - \bar{\alpha}^m)(\bar{u}^{m+1} - \bar{u}^m)(e_h^{m+1} - e_h^m)\Big| \\
&\le c\int_{\Omega_h}\Big|\frac{\beta(\nu_h^m)}{Q_h^m} - \frac{\beta(\bar{\nu}^m)}{\bar{Q}^m}\Big||e_h^{m+1} - e_h^m| \\
&\le c\int_{\Omega_h}|\nu_h^m - \bar{\nu}^m|(|\mathrm{e}^{m+1} - \mathrm{e}^m| + |\varepsilon^{m+1} - \varepsilon^m|) \\
&\le \frac{\delta}{\tau}\int_{\Omega_h}\frac{|\mathrm{e}^{m+1} - \mathrm{e}^m|^2}{Q_h^m} + \frac{1}{\tau}\int_{\Omega_h}|\varepsilon^{m+1} - \varepsilon^m|^2 \\
&\quad + \frac{c}{\delta}\tau\int_{\Omega_h}|\bar{\nu}_h^m - \nu^m|^2 Q_h^m \\
&\le \frac{\delta}{\tau}\int_{\Omega_h}\frac{|\mathrm{e}^{m+1} - \mathrm{e}^m|^2}{Q_h^m} + ch^4\int_{m\tau}^{(m+1)\tau}\|\bar{u}_t\|^2_{H^2(\Omega_h)} \\
&\quad + \frac{c}{\delta}\tau\int_{\Omega_h}|\bar{\nu}^m - \nu_h^m|^2 Q_h^m.
\end{aligned}$$

For the second term in (4.8) we have (ignoring the factor λ)

$$\begin{aligned}
&\int_{\Omega_h}\frac{\gamma(\nu_h^m)}{Q_h^m}\nabla(\bar{u}^{m+1} - \bar{u}^m)\cdot\nabla(e_h^{m+1} - e_h^m) \\
&= \int_{\Omega_h}\frac{\gamma(\bar{\nu}^m)}{\bar{Q}^m}\nabla(\bar{u}^{m+1} - \bar{u}^m)\cdot\nabla(e_h^{m+1} - e_h^m) \\
&\quad + \int_{\Omega_h}\Big(\frac{\gamma(\nu_h^m)}{Q_h^m} - \frac{\gamma(\bar{\nu}^m)}{\bar{Q}^m}\Big)\nabla(\bar{u}^{m+1} - \bar{u}^m)\cdot\nabla(e_h^{m+1} - e_h^m) \\
&=: I_1 + I_2.
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
I_1 = &-\int_{\Omega_h}\nabla\Big(\frac{\gamma(\bar{\nu}^m)}{\bar{Q}^m}\Big)\cdot\nabla(\bar{u}^{m+1} - \bar{u}^m)(e_h^{m+1} - e_h^m) \\
&-\int_{\Omega_h}\frac{\gamma(\nu^m)}{Q_h^m}\Delta(\bar{u}^{m+1} - \bar{u}^m)(\mathrm{e}_h^{m+1} - \mathrm{e}_h^m) \\
&-\int_{\Omega_h}\gamma(\bar{\nu}^m)\Big(\frac{1}{\bar{Q}^m} - \frac{1}{Q_h^m}\Big)\Delta(\bar{u}^{m+1} - \bar{u}^m)(\mathrm{e}_h^{m+1} - \mathrm{e}_h^m).
\end{aligned}$$

Using (4.1), Young's inequality, (3.1) and (4.7) we obtain

$$|I_1| \le c\tau\int_{\Omega_h}|\mathrm{e}_h^{m+1} - \mathrm{e}_h^m| + c\int_{\Omega_h}|\Delta(\bar{u}^{m+1} - \bar{u}^m)|\frac{|\mathrm{e}_h^{m+1} - \mathrm{e}_h^m|}{Q_h^m}$$

$$
\begin{aligned}
&+ \quad c\int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|\,|\Delta(\bar{u}^{m+1} - \bar{u}^m)|\,|\mathrm{e}_h^{m+1} - \mathrm{e}_h^m| \\
&\leq \frac{\delta}{\tau}\int_{\Omega_h} \frac{|\mathrm{e}_h^{m+1} - \mathrm{e}_h^m|^2}{Q_h^m} + \frac{c}{\delta}\tau^3\int_{\Omega_h} Q_h^m + \frac{c}{\delta}\tau^2\int_{m\tau}^{(m+1)\tau} \|\bar{u}_t\|^2_{H^2(\Omega_h)} \\
&+ \quad \frac{c}{\delta}\|D^2\bar{u}\|^2_{L^\infty(\Omega_h\times(0,T))}\tau\int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|^2 Q_h^m \\
&\leq \frac{\delta}{\tau}\int_{\Omega_h} \frac{|\mathrm{e}^{m+1} - \mathrm{e}^m|^2}{Q_h^m} + \frac{c}{\delta}\tau^3 + \frac{c}{\delta}\tau\int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|^2 Q_h^m \\
&+ \frac{c}{\delta}(\tau^2 + h^4)\int_{m\tau}^{(m+1)\tau} \|\bar{u}_t\|^2_{H^2(\Omega_h)}.
\end{aligned}
$$

Furthermore, again by (4.1)

$$
\begin{aligned}
|I_2| &\leq c\tau\int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|\,|\nabla(e_h^{m+1} - e_h^m)| \\
&\leq \tau\int_{\Omega_h} \frac{|\nabla(\mathrm{e}^{m+1} - \mathrm{e}^m)|^2}{Q_h^m} + \tau\int_{\Omega_h} |\nabla(\varepsilon^{m+1} - \varepsilon^m)|^2 \\
&\quad + c\tau\int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|^2 Q_h^m \\
&\leq \tau\int_{\Omega_h} \frac{|\nabla(\mathrm{e}^{m+1} - \mathrm{e}^m)|^2}{Q_h^m} + c\tau^2 h^2\int_{m\tau}^{(m+1)\tau} \|\bar{u}_t\|^2_{H^2(\Omega_h)} \\
&\quad + c\tau\int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|^2 Q_h^m.
\end{aligned}
$$

Next, Young's inequality, (4.4), (4.5) and (4.7) imply

$$
\begin{aligned}
&\Big|\int_{\Omega_h} \bar{\alpha}^m S^m (e_h^{m+1} - e_h^m)\Big| \\
&\leq \int_{\Omega_h} |\bar{\alpha}^m - \alpha_h^m|\,|S^m|\,|\mathrm{e}_h^{m+1} - \mathrm{e}_h^m| + \int_{\Omega_h} |\alpha_h^m|\,|S^m|\,|\mathrm{e}_h^{m+1} - \mathrm{e}_h^m| \\
&\leq c\|S^m\|_{L^\infty}\int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|\,|\mathrm{e}_h^{m+1} - \mathrm{e}_h^m| + c\int_{\Omega_h} |S^m|\,\frac{|\mathrm{e}_h^{m+1} - \mathrm{e}_h^m|}{Q_h^m} \\
&\leq c\int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|\,|\mathrm{e}_h^{m+1} - \mathrm{e}_h^m| + c\int_{\Omega_h} |S^m|\,\frac{|\mathrm{e}_h^{m+1} - \mathrm{e}_h^m|}{Q_h^m} \\
&\leq \frac{\delta}{\tau}\int_{\Omega_h} \frac{|\mathrm{e}^{m+1} - \mathrm{e}^m|^2}{Q_h^m} + \frac{\delta}{\tau}\int_{\Omega} |\varepsilon^{m+1} - \varepsilon^m|^2 \\
&\quad + \frac{c}{\delta}\tau\int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|^2 Q_h^m + \frac{c}{\delta}\tau\|S^m\|^2
\end{aligned}
$$

$$\leq \frac{\delta}{\tau}\int_{\Omega_h}\frac{|\mathrm{e}^{m+1}-\mathrm{e}^m|^2}{Q_h^m}+\frac{c}{\delta}\tau\int_{\Omega_h}|\bar\nu^m-\nu_h^m|^2Q_h^m+\frac{c}{\delta}\tau h^2$$
$$+\frac{c}{\delta}(\tau^2+h^4)\int_{m\tau}^{(m+1)\tau}(\|\bar u_{tt}\|^2+\|\bar u_t\|^2_{H^2(\Omega_h)})ds.$$

Young's inequality and (4.7) imply

$$\Big|\frac{1}{\tau}\int_{\Omega_h}\alpha_h^m(\mathrm{e}^{m+1}-\mathrm{e}^m)(\varepsilon^{m+1}-\varepsilon^m)\Big|$$
$$\leq \frac{c}{\tau}\int_{\Omega_h}\frac{1}{Q_h^m}|\mathrm{e}^{m+1}-\mathrm{e}^m|\,|\varepsilon^{m+1}-\varepsilon^m|$$
$$\leq \frac{\delta}{\tau}\int_{\Omega_h}\frac{|\mathrm{e}^{m+1}-\mathrm{e}^m|^2}{Q_h^m}+\frac{c}{\delta\tau}\int_{\Omega_h}|\varepsilon^{m+1}-\varepsilon^m|^2$$
$$\leq \frac{\delta}{\tau}\int_{\Omega_h}\frac{|\mathrm{e}^{m+1}-\mathrm{e}^m|^2}{Q_h^m}+\frac{c}{\delta}h^4\int_{m\tau}^{(m+1)\tau}\|\bar u_t\|^2_{H^2(\Omega_h)}ds,$$

and using similar arguments we finally obtain

$$|\sum_{i=1}^n\int_{\Omega_h}(\gamma_{p_i}(\bar\nu^m)-\gamma_{p_i}(\nu_h^m))(\varepsilon^{m+1}-\varepsilon^m)_{x_i}|$$
$$+|\lambda\int_{\Omega_h}\frac{\gamma(\nu_h^m)}{Q_h^m}\nabla(\mathrm{e}^{m+1}-\mathrm{e}^m)\cdot\nabla(\varepsilon^{m+1}-\varepsilon^m)|$$
$$\leq c\int_{\Omega_h}|\bar\nu^m-\nu_h^m|\,|\nabla(\varepsilon^{m+1}-\varepsilon^m)|$$
$$+c\int_{\Omega_h}\frac{1}{Q_h^m}|\nabla(\mathrm{e}^{m+1}-\mathrm{e}^m)|\,|\nabla(\varepsilon^{m+1}-\varepsilon^m)|$$
$$\leq \tau\int_{\Omega_h}|\bar\nu^m-\nu_h^m|^2Q_h^m+\tau\int_{\Omega_h}\frac{|\nabla(\mathrm{e}^{m+1}-\mathrm{e}^m)|^2}{Q_h^m}$$
$$+\frac{c}{\tau}\int_{\Omega_h}|\nabla(\varepsilon^{m+1}-\varepsilon^m)|^2$$
$$\leq \tau\int_{\Omega_h}|\bar\nu^m-\nu_h^m|^2Q_h^m+\tau\int_{\Omega_h}\frac{|\nabla(\mathrm{e}^{m+1}-\mathrm{e}^m)|^2}{Q_h^m}$$
$$+ch^2\int_{m\tau}^{(m+1)\tau}\|\bar u_t\|^2_{H^2(\Omega_h)}ds.$$

If we insert the above estimates into (4.8) and choose δ small (recalling (2.6)), we obtain with the help of Lemma 4.2

$$\frac{1}{2\tau}\int_{\Omega_h}\alpha_h^m|\mathrm{e}^{m+1}-\mathrm{e}^m|^2+D^{m+1}-D^m$$

$$
\begin{aligned}
&+(\lambda \inf_{|p|=1} \gamma(p) - \bar{\gamma} - c\tau) \int_{\Omega_h} \frac{|\nabla(\mathrm{e}^{m+1} - \mathrm{e}^m)|^2}{Q_h^m} \\
\le\ & c\tau(\tau^2 + h^2) \\
&+c\tau\Big(\int_{\Omega_h} |\bar{\nu}^{m+1} - \nu_h^{m+1}|^2 Q_h^{m+1} + \int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|^2 Q_h^m\Big) \\
&+c(\tau^2 + h^2) \int_{m\tau}^{(m+1)\tau} (\|\bar{u}_{tt}(s)\|^2 + \|\bar{u}_t\|^2_{H^2(\Omega_h)}) ds \\
\le\ & c(\tau^2 + h^2)\Big(\tau + \int_{m\tau}^{(m+1)\tau} (\|\bar{u}_{tt}(s)\|^2 + \|\bar{u}_t\|^2_{H^2(\Omega_h)}) ds\Big) \\
&+c\tau(D^{m+1} + D^m)
\end{aligned}
$$

by Lemma 4.1. Summation from $m = 0, ..., M-1$ yields in view of (4.1)

$$
\begin{aligned}
D^M + \sum_{m=0}^{M-1} \Big(&\frac{1}{2\tau} \int_{\Omega_h} \alpha_h^m |\mathrm{e}^{m+1} - \mathrm{e}^m|^2 \\
&+ (\lambda \inf_{|p|=1} \gamma(p) - \bar{\gamma} - c\tau) \int_{\Omega_h} \frac{|\nabla(\mathrm{e}^{m+1} - \mathrm{e}^m)|^2}{Q_h^m}\Big) \\
&\le c(\tau^2 + h^2) + c\tau \sum_{m=0}^{M-1} D^m + c\tau D^M
\end{aligned}
$$

which together with (1.12) implies

$$
D^M \le c(\tau^2 + h^2) + c\tau \sum_{m=0}^{M-1} D^m, \qquad 1 \le M \le [\frac{T}{\tau}],
$$

provided that τ is sufficiently small. From the discrete Gronwall lemma and (2.6) we infer

$$
\begin{aligned}
&\sum_{m=0}^{M-1} \frac{1}{\tau} \int_{\Omega_h} \frac{|\mathrm{e}^{m+1} - \mathrm{e}^m|^2}{Q_h^m} + \max_{0 \le m \le M} D^m \\
&\le\ c(\tau^2 + h^2)\mathrm{e}^{cM\tau} \le c(\tau^2 + h^2)\mathrm{e}^{cT}, \quad 0 \le M \le [\frac{T}{\tau}]. \qquad (4.15)
\end{aligned}
$$

Thus we can formulate our main result,

Theorem 4.3 *Suppose that (1.12) holds. Then there exists* $\tau_0 > 0$ *such that for all* $0 < \tau \le \tau_0$

$$
\begin{aligned}
&\sum_{m=0}^{[\frac{T}{\tau}]-1} \tau \int_{\Omega \cap \Omega_h} |V^m - V_h^m|^2 Q_h^m \\
&\quad + \max_{0 \le m \le [\frac{T}{\tau}]} \int_{\Omega \cap \Omega_h} |\nu^m - \nu_h^m|^2 Q_h^m \le c(\tau^2 + h^2).
\end{aligned}
$$

Proof. In view of (4.15) and Lemma 4.1 we only have to prove the bound for the normal velocity. To begin,

$$\bar{V}^m - V_h^m = -\frac{\bar{u}_t(\cdot, m\tau)}{\bar{Q}^m} + \frac{(u_h^{m+1} - u_h^m)/\tau}{Q_h^m}$$
$$= -\frac{(\mathrm{e}^{m+1} - \mathrm{e}^m)/\tau}{Q_h^m} + \frac{\bar{u}^{m+1} - \bar{u}^m}{\tau}\Big(\frac{1}{Q_h^m} - \frac{1}{\bar{Q}^m}\Big) + \frac{S^m}{\bar{Q}^m}.$$

Using Lemma 4.1 and (4.15) we obtain

$$\sum_{m=0}^{[\frac{T}{\tau}]-1} \tau \int_{\Omega_h} |\bar{V}^m - V_h^m|^2 Q_h^m \leq \sum_{m=0}^{[\frac{T}{\tau}]-1} \Big(\frac{1}{\tau}\int_{\Omega_h} \frac{|\mathrm{e}^{m+1} - \mathrm{e}^m|^2}{Q_h^m}$$
$$+\tau \int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|^2 Q_h^m + \tau \int_{\Omega_h} |S^m|^2 \frac{Q_h^m}{(\bar{Q}^m)^2}\Big)$$
$$\leq c(\tau^2 + h^2) + \sum_{m=0}^{[\frac{T}{\tau}]-1} \tau \int_{\Omega_h} |S^m|^2 \frac{Q_h^m}{(\bar{Q}^m)^2}.$$

In order to estimate the last term we note that $\frac{Q_h^m}{(\bar{Q}^m)^2} \leq \frac{Q_h^m}{\bar{Q}^m} \leq 1 + |\bar{\nu}^m - \nu_h^m| Q_h^m$ which together with (4.4), (4.5) and (4.15) implies

$$\sum_{m=0}^{[\frac{T}{\tau}]-1} \tau \int_{\Omega_h} |S^m|^2 \frac{Q_h^m}{(\bar{Q}^m)^2}$$
$$\leq 2\sum_{m=0}^{[\frac{T}{\tau}]-1} \tau \int_{\Omega_h} \Big(|S^m|^2 + |S^m|^2 |\bar{\nu}^m - \nu_h^m|^2 Q_h^m\Big)$$
$$\leq c\sum_{m=0}^{[\frac{T}{\tau}]-1} \tau^2 \int_{m\tau}^{(m+1)\tau} \|\bar{u}_{tt}\|^2 ds + ch^2 + c\sum_{m=0}^{[\frac{T}{\tau}]-1} \tau \int_{\Omega_h} |\bar{\nu}^m - \nu_h^m|^2 Q_h^m$$
$$\leq c(\tau^2 + h^2).$$ □

5. Numerical tests

In order to check the estimates which were proved in Theorem 3.1 and Theorem 4.3 we computed test examples for the convergence estimate and for the stability estimate.

A test for the convergence is derived from the fact that the Wulff shape shrinks homothetically during the evolution. We have chosen the very strong anisotropy

$$\gamma(p) = \sqrt{0.01 p_1^2 + p_2^2 + p_3^2}.$$

Table 1. Absolute errors for the test problem with $\tau = 0.01h$

h	$error(\nu)$	eoc	$error(V)$	eoc	$L^\infty(H^1)$	eoc	$L^\infty(L^2)$	eoc
7.0711e-2	9.7027e-2	–	2.3278e-2	–	1.3366e-1	–	3.1700e-3	-
3.5355e-2	2.3213e-2	2.06	6.0827e-3	1.94	4.4935e-2	1.57	7.3639e-4	2.11
2.6050e-2	2.4818e-2	-0.22	7.5203e-3	-0.70	4.5372e-2	-0.03	3.0728e-4	2.86
1.4861e-2	1.3868e-2	1.04	4.1117e-3	1.08	2.4163e-2	1.12	8.8373e-5	2.22
7.8462e-3	7.0232e-3	1.07	1.9806e-3	1.14	1.2256e-2	1.06	2.7864e-5	1.81
4.0210e-3	3.5368e-3	1.03	1.0176e-3	1.00	6.1725e-3	1.03	1.2378e-5	1.22
2.0342e-3	1.8103e-3	0.98	5.2675e-4	0.97	3.1225e-3	1.00	6.3199e-6	0.99
1.0229e-3	9.2938e-4	0.97	2.6988e-4	0.97	1.5799e-3	0.99	3.2715e-6	0.96

If we denote the dual of the anisotropy γ by γ^*,

$$\gamma^*(p) = \sup_{|q|=1} \frac{\langle p, q\rangle}{\gamma(q)},$$

then the equation

$$\gamma^*(x, u(x,t)) = \sqrt{1-4t}$$

defines a solution of the differential equation

$$u_t - \gamma(\nabla u, -1) \sum_{i,j=1}^{2} \gamma_{p_i p_j}(\nabla u, -1) u_{x_i x_j} = 0$$

on the domain $\Omega = \{x \in \mathbb{R}^2 \mid |x| < R = 0.035355\}$ for $t \in (0, 0.125)$. Thus the mobility is chosen as $\beta = 1/\gamma$. The exact solution is given by $u(x,t) = \sqrt{1 - 4t - 100x_1^2 - x_2^2}$. The condition (1.12) is easily checked for this anisotropy and it is satisfied, if we choose the parameter $\lambda = 81.0$. Since the quantity $\rho = \lambda \inf_{S^n} \gamma - \overline{\gamma}$ then is a small number, it is necessary to choose the time step $\tau \le \tau_0$ with a relatively small τ_0. We use $\tau = 0.01h$ as a uniform time step size. A coupling of time step size τ and grid size h is necessary for the test computations only. The asymptotic order of the errors then is not spoiled, see Theorem 4.3. Table 1 shows the grid size h, the errors

$$error(V) = \left(\sum_{m=0}^{M} \tau \int_{\Omega_h} |V^m - V_h^m|^2 Q_h^m \right)^{\frac{1}{2}},$$

$$error(\nu) = \left(\max_{0\le m\le M} \int_{\Omega_h} |\nu^m - \nu_h^m|^2 Q_h^m \right)^{\frac{1}{2}},$$

Table 2. Absolute errors for the test problem with $\tau = 0.5h$

h	$error(\nu)$	eoc	$error(V)$	eoc	$L^\infty(H^1)$	eoc	$L^\infty(L^2)$	eoc
6.e-2	7.8945e-2	–	2.6047e-2	–	1.0669e-1	–	2.1472e-3	-
3.e-2	2.3428e-2	1.75	4.7939e-3	2.44	4.1133e-2	1.38	6.8904e-4	1.64
2.2104e-2	2.2062e-2	0.20	3.2553e-3	1.27	3.4393e-2	0.59	3.4702e-4	2.25
1.261e-2	1.3073e-2	0.93	2.2360e-3	0.67	1.9276e-2	1.03	1.7314e-4	1.24
6.6578e-3	8.4522e-3	0.68	1.5444e-3	0.58	1.3128e-2	0.60	1.2520e-4	0.51
3.4120e-3	6.3062e-3	0.44	1.2419e-3	0.14	1.0222e-2	0.11	9.7372e-5	0.38
1.7261e-3	4.8719e-3	0.38	1.0307e-3	0.27	7.9901e-3	0.15	7.1246e-5	0.46

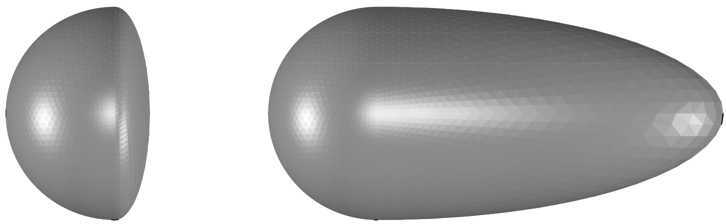

Fig. 1. Frank diagram $\mathcal{F}$ (left) and Wulff shape $\mathcal{W}$ (right) for the anisotropy (1.2)

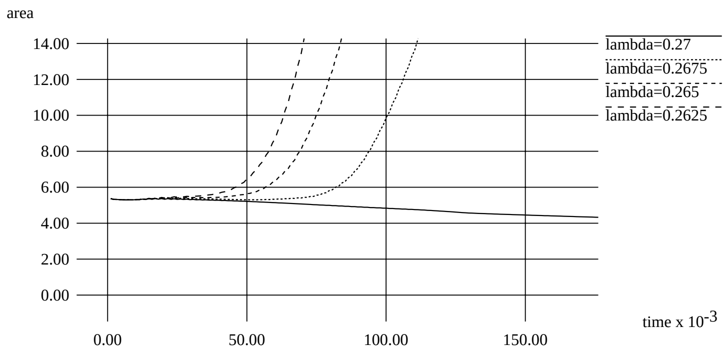

Fig. 2. Weighted area as a function of time for different values of the parameter λ

and the corresponding experimental orders of convergence (eoc) between two succesive grid sizes. We add two columns with the errors in $L^\infty((0,T),H^1(\Omega))$ and $L^\infty((0,T),L^2(\Omega))$. Obviously the results of the asymptotic error estimates of Theorem 4.3 are reproduced in our test computations. That a condition of the form $\tau \le \tau_0$ is necessary for small ρ can be seen from Table 2, where we have chosen $\tau = 0.5h$.

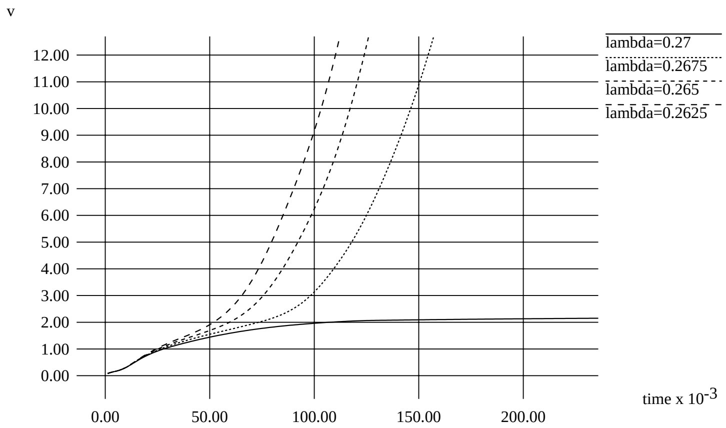

Fig. 3. v as a function of time for different values of the parameter λ

A stability result for the natural norms

$$v(m) = \left(\tau \sum_{k=1}^{m-1} \int_{\Omega_h} \beta(\nu_h^k)\, |V_h^k|^2 Q_h^k \right)^{\frac{1}{2}}, \quad area(m) = \int_{\Omega_h} \gamma(\nu_h^m) Q_h^m \tag{5.1}$$

follows from Theorem 3.1 under the condition that the parameter λ satisfies the condition $\lambda \inf_{S^n} \gamma - \overline{\gamma} \geq 0$. In particular, from the stability estimate in Theorem 3.1 it is obvious that the quantity $area$ is a decreasing function of the time step m, $m = 1, \ldots, M$. Since it is not known, if this condition is optimal, we computed the quantities (5.1) for different values of λ. These values are visualized in Fig. 2 and Fig. 3 as a function of time. The tests show, that a condition on the size of the parameter λ is necessary for stability. Here we did not use an exact solution but only provided the initial data.

The most interesting examples for our problem are anisotropies which are close to crystalline weights. In Fig. 4 and 5 we show the results of a long time computation for an anisotropy function which is a regularized version of $\gamma(p) = |p|_{l^\infty}$ so that the Frank diagram is a cube and the Wulff shape is an octahedron. The regularization is made such that the Frank diagram becomes strictly convex and smooth. We have chosen a non zero constant right hand side for the equation. In Fig. 4 we show the level lines of the initial function, of four time steps and of the stationary solution. The boundary values were kept fixed during the evolution. We can see that the octahedral shape develops during the evolution. In Fig. 5 we show the initial graph and the stationary graph. The domain was the unit disk.

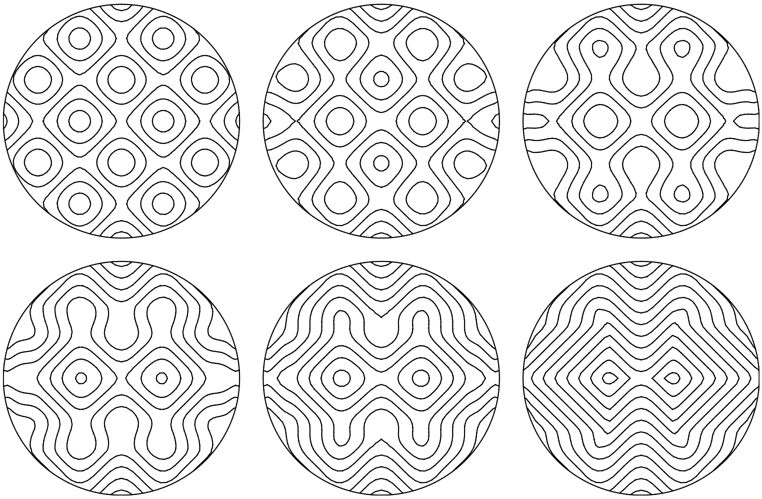

Fig. 4. Level lines for the time steps $0, 250, 500, 750, 1000, 3000$ for a regularized crystalline anisotropy

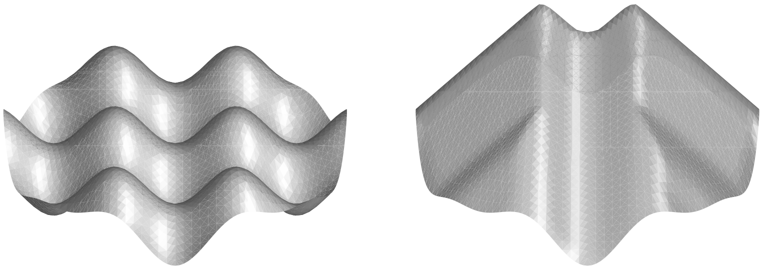

Fig. 5. Initial value and stationary solution for a regularized crystalline anisotropy

6. Appendix

Lemma 6.1 *Let u and v be in $H^{1,\infty}(\Omega)$ and assume that $|\nabla u| \le K$ a.e. in Ω. Then there exists a constant $c_0 = c_0(K) > 0$ such that*

$$|s\nu(u) + (1-s)\nu(v)| \ge c_0 \qquad \textit{a.e. in } \Omega, \;\; \textit{for all } s \in [0,1].$$

Proof. We have for all $s \in [0,1]$ by (1.4)

$$\begin{aligned}
&|s\nu(u) + (1-s)\nu(v)|^2 \\
&= |s\frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}} + (1-s)\frac{(\nabla v, -1)}{\sqrt{1+|\nabla v|^2}}|^2 \\
&= s^2 + (1-s)^2 + 2s(1-s)\frac{\nabla u \cdot \nabla v + 1}{\sqrt{1+|\nabla u|^2}\sqrt{1+|\nabla v|^2}}
\end{aligned}$$

$$\geq 1 - 2s(1-s) + 2s(1-s)\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \frac{\nabla v}{\sqrt{1+|\nabla v|^2}}$$
$$\geq 1 - 2s(1-s)\Big(1 + \frac{K}{\sqrt{1+K^2}}\Big)$$
$$\geq 1 - \frac{1}{2}\Big(1 + \frac{K}{\sqrt{1+K^2}}\Big),$$

which implies the result. □

Remark 6.2 In a typical application of this result u will be the continuous and v the discrete solution, for which no uniform gradient bound is available.

Lemma 6.3 *Let u and v be in $H^{1,\infty}(\Omega)$. Then we have a.e. in Ω:*

(A.1) $|Q(u) - Q(v)| \leq |\nu(u) - \nu(v)|Q(u)Q(v)$

(A.2) $|\nabla(u-v)|^2 = (Q(u) - Q(v))^2 + |\nu(u) - \nu(v)|^2 Q(u)Q(v)$

Proof. The first relation is a consequence of the fact that $\frac{-1}{Q(u)}$ and $\frac{-1}{Q(v)}$ are the last components of $\nu(u), \nu(v)$ respectively, while (A.2) follows from an elementary calculation. □

Lemma 6.4

$$|(\bar{\nu}^{m+1} - \nu_h^{m+1}) - (\bar{\nu}^m - \nu_h^m)|^2 Q_h^{m+1} Q_h^m$$
$$\leq (1 + c\tau)|\nabla(\mathrm{e}^{m+1} - \mathrm{e}^m)|^2 + c\tau Q_h^{m+1} Q_h^m (\tau^2 + |\bar{\nu}^{m+1} - \nu_h^{m+1}|^2).$$

Proof. Using (A.2) and the relation $(\nabla\bar{u}, -1) = Q(\bar{u})\nu(\bar{u})$ we derive

$$|\nabla(\mathrm{e}^{m+1} - \mathrm{e}^m)|^2 = |\nabla(\bar{u}^{m+1} - \bar{u}^m) - \nabla(u_h^{m+1} - u_h^m)|^2$$
$$= |\nabla(\bar{u}^{m+1} - \bar{u}^m)|^2 - 2\,\nabla(\bar{u}^{m+1} - \bar{u}^m) \cdot \nabla(u_h^{m+1} - u_h^m)$$
$$\quad + |\nabla(u_h^{m+1} - u_h^m)|^2$$
$$= (\bar{Q}^{m+1} - \bar{Q}^m)^2 + |\bar{\nu}^{m+1} - \bar{\nu}^m|^2 \bar{Q}^{m+1}\bar{Q}^m$$
$$\quad - 2\langle(\bar{Q}^{m+1}\bar{\nu}^{m+1} - \bar{Q}^m\bar{\nu}^m), (Q_h^{m+1}\nu_h^{m+1} - Q_h^m\nu_h^m)\rangle$$
$$\quad + (Q_h^{m+1} - Q_h^m)^2 + |\nu_h^{m+1} - \nu_h^m|^2 Q_h^{m+1} Q_h^m$$
$$= \Big((\bar{Q}^{m+1} - \bar{Q}^m) - (Q_h^{m+1} - Q_h^m)\Big)^2$$
$$\quad + 2(\bar{Q}^{m+1} - \bar{Q}^m)(Q_h^{m+1} - Q_h^m) \tag{A.3}$$
$$\quad + |(\bar{\nu}^{m+1} - \bar{\nu}^m) - (\nu_h^{m+1} - \nu_h^m)|^2 Q_h^{m+1} Q_h^m$$
$$\quad + |\bar{\nu}^{m+1} - \bar{\nu}^m|^2 (\bar{Q}^{m+1}\bar{Q}^m - Q_h^{m+1} Q_h^m)$$
$$\quad + 2\langle(\bar{\nu}^{m+1} - \bar{\nu}^m), (\nu_h^{m+1} - \nu_h^m)\rangle Q_h^{m+1} Q_h^m$$
$$\quad - 2\langle \bar{Q}^{m+1}\bar{\nu}^{m+1} - \bar{Q}^m\bar{\nu}^m, Q_h^{m+1}\nu_h^{m+1} - Q_h^m\nu_h^m\rangle\ .$$

Let us look at the last term separately. Clearly,

$$\begin{aligned}
&\langle \bar{Q}^{m+1}\bar{\nu}^{m+1} - \bar{Q}^m\bar{\nu}^m, Q_h^{m+1}\nu_h^{m+1} - Q_h^m\nu_h^m\rangle \\
&= \langle \bar{Q}^{m+1}\bar{\nu}^{m+1} - \bar{Q}^m\bar{\nu}^m, \bar{\nu}^{m+1}\rangle(Q_h^{m+1} - Q_h^m) \\
&\quad + \langle \bar{Q}^{m+1}\bar{\nu}^{m+1} - \bar{Q}^m\bar{\nu}^m, \nu_h^{m+1} - \bar{\nu}^{m+1}\rangle(Q_h^{m+1} - Q_h^m) \\
&\quad + \langle \bar{Q}^{m+1}\bar{\nu}^{m+1} - \bar{Q}^m\bar{\nu}^m, \nu_h^{m+1} - \nu_h^m\rangle Q_h^m \\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

Note first that

$$\begin{aligned}
A_1 &= \bar{Q}^{m+1}(Q_h^{m+1} - Q_h^m) - \langle \bar{\nu}^m, \bar{\nu}^{m+1}\rangle \bar{Q}^m(Q_h^{m+1} - Q_h^m) \\
&= (\bar{Q}^{m+1} - \bar{Q}^m)(Q_h^{m+1} - Q_h^m) \\
&\quad + (1 - \langle \bar{\nu}^m, \bar{\nu}^{m+1}\rangle)\bar{Q}^m(Q_h^{m+1} - Q_h^m) \\
&= (\bar{Q}^{m+1} - \bar{Q}^m)(Q_h^{m+1} - Q_h^m) \\
&\quad + \frac{1}{2}|\bar{\nu}^{m+1} - \bar{\nu}^m|^2\bar{Q}^m(Q_h^{m+1} - Q_h^m).
\end{aligned}$$

Furthermore, observing that

$$\begin{aligned}
&\bar{Q}^{m+1}\bar{\nu}^{m+1} - \bar{Q}^m\bar{\nu}^m \\
&= Q_h^{m+1}(\bar{\nu}^{m+1} - \bar{\nu}^m) + (\bar{Q}^{m+1} - Q_h^{m+1})(\bar{\nu}^{m+1} - \bar{\nu}^m) \\
&\quad + (\bar{Q}^{m+1} - \bar{Q}^m)\bar{\nu}^m
\end{aligned}$$

as well as $\langle \bar{\nu}^m, \nu_h^{m+1} - \nu_h^m\rangle = \langle \bar{\nu}^m - \frac{1}{2}(\nu_h^{m+1} + \nu_h^m), \nu_h^{m+1} - \nu_h^m\rangle$ we obtain

$$\begin{aligned}
A_3 &= \langle \bar{\nu}^{m+1} - \bar{\nu}^m, \nu_h^{m+1} - \nu_h^m\rangle Q_h^{m+1}Q_h^m \\
&\quad + \langle \bar{\nu}^{m+1} - \bar{\nu}^m, \nu_h^{m+1} - \nu_h^m\rangle(\bar{Q}^{m+1} - Q_h^{m+1})Q_h^m \\
&\quad + \langle \bar{\nu}^m - \frac{1}{2}(\nu_h^{m+1} + \nu_h^m), \nu_h^{m+1} - \nu_h^m\rangle(\bar{Q}^{m+1} - \bar{Q}^m)Q_h^m.
\end{aligned}$$

In conclusion,

$$\begin{aligned}
&\langle \bar{Q}^{m+1}\bar{\nu}^{m+1} - \bar{Q}^m\bar{\nu}^m, Q_h^{m+1}\nu_h^{m+1} - Q_h^m\nu_h^m\rangle \\
&= (\bar{Q}^{m+1} - \bar{Q}^m)(Q_h^{m+1} - Q_h^m) + \frac{1}{2}|\bar{\nu}^{m+1} - \bar{\nu}^m|^2\bar{Q}^m(Q_h^{m+1} - Q_h^m) \\
&\quad + \langle \bar{Q}^{m+1}\bar{\nu}^{m+1} - \bar{Q}^m\bar{\nu}^m, \nu_h^{m+1} - \bar{\nu}^{m+1}\rangle(Q_h^{m+1} - Q_h^m) \\
&\quad + \langle \bar{\nu}^{m+1} - \bar{\nu}^m, \nu_h^{m+1} - \nu_h^m\rangle Q_h^{m+1}Q_h^m \\
&\quad + \langle \bar{\nu}^{m+1} - \bar{\nu}^m, \nu_h^{m+1} - \nu_h^m\rangle(\bar{Q}^{m+1} - Q_h^{m+1})Q_h^m \\
&\quad + \langle \bar{\nu}^m - \frac{1}{2}(\nu_h^{m+1} + \nu_h^m), \nu_h^{m+1} - \nu_h^m\rangle(\bar{Q}^{m+1} - \bar{Q}^m)Q_h^m
\end{aligned}$$

so that we obtain by inserting this expression into (A.3)

$$\begin{aligned}
&|\nabla(\mathrm{e}^{m+1}-\mathrm{e}^{m})|^2 \\
&= \Big((\bar{Q}^{m+1}-\bar{Q}^{m})-(Q_h^{m+1}-Q_h^{m})\Big)^2 \\
&\quad +|(\bar{\nu}^{m+1}-\bar{\nu}^{m})-(\nu_h^{m+1}-\nu_h^{m})|^2 Q_h^{m+1}Q_h^{m} \\
&\quad +|\bar{\nu}^{m+1}-\bar{\nu}^{m}|^2(\bar{Q}^{m+1}\bar{Q}^{m}-Q_h^{m+1}Q_h^{m}) \\
&\quad -|\bar{\nu}^{m+1}-\bar{\nu}^{m}|^2\bar{Q}^{m}(Q_h^{m+1}-Q_h^{m}) \\
&\quad -2\langle \bar{Q}^{m+1}\bar{\nu}^{m+1}-\bar{Q}^{m}\bar{\nu}^{m},\nu_h^{m+1}-\bar{\nu}^{m+1}\rangle(Q_h^{m+1}-Q_h^{m}) \\
&\quad -2\langle \bar{\nu}^{m+1}-\bar{\nu}^{m},\nu_h^{m+1}-\nu_h^{m}\rangle(\bar{Q}^{m+1}-Q_h^{m+1})Q_h^{m} \\
&\quad -2\langle \bar{\nu}^{m}-\frac{1}{2}(\nu_h^{m+1}+\nu_h^{m}),\nu_h^{m+1}-\nu_h^{m}\rangle(\bar{Q}^{m+1}-\bar{Q}^{m})Q_h^{m}.
\end{aligned}$$

In view of the regularity of $\bar{u}$ and (A.1)

$$\begin{aligned}
&|\nabla(\mathrm{e}^{m+1}-\mathrm{e}^{m})|^2 \\
&\geq |(\bar{\nu}^{m+1}-\bar{\nu}^{m})-(\nu_h^{m+1}-\nu_h^{m})|^2 Q_h^{m+1}Q_h^{m} \\
&\quad -c\tau^2|\bar{Q}^{m+1}\bar{Q}^{m}-Q_h^{m+1}Q_h^{m}| \\
&\quad -c\tau^2|\nu_h^{m+1}-\nu_h^{m}|Q_h^{m+1}Q_h^{m} \\
&\quad -c\tau|\bar{\nu}^{m+1}-\nu_h^{m+1}|\,|\nu_h^{m+1}-\nu_h^{m}|Q_h^{m+1}Q_h^{m} \\
&\quad -c\tau(\tau+|\bar{\nu}^{m}-\nu_h^{m}|+|\bar{\nu}^{m+1}-\nu_h^{m+1}|)|\nu_h^{m+1}-\nu_h^{m}|Q_h^{m}. \qquad \text{(A.4)}
\end{aligned}$$

Again by (A.1),

$$\begin{aligned}
&|\bar{Q}^{m+1}\bar{Q}^{m}-Q_h^{m+1}Q_h^{m}| \leq |\bar{Q}^{m+1}-Q_h^{m+1}|\bar{Q}^{m}+|\bar{Q}^{m}-Q_h^{m}|Q_h^{m+1} \\
&\leq c|\bar{\nu}^{m+1}-\nu_h^{m+1}|Q_h^{m+1}+c|\bar{\nu}^{m}-\nu_h^{m}|Q_h^{m+1}Q_h^{m}.
\end{aligned}$$

Inserting this inequality into (A.4) and using (4.11) we finally obtain

$$\begin{aligned}
|\nabla(\mathrm{e}^{m+1}-\mathrm{e}^{m})|^2 \geq (1-c\tau)|(\bar{\nu}^{m+1}-\nu_h^{m+1})-(\bar{\nu}^{m}-\nu_h^{m})|^2 Q_h^{m+1}Q_h^{m} \\
-c\tau Q_h^{m+1}Q_h^{m}(\tau^2+|\bar{\nu}^{m+1}-\nu_h^{m+1}|^2)
\end{aligned}$$

which implies the result.

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