# A new stable time discretization of higher order for evolution equations in a Hilbert space

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# Minimizing an energy functional

- TASK : minimize **energy functional** : $W : V \to \mathbb{R}^+$
- assume $\exists$ **derivative** : $W' : V \to V'$

$$\left\langle W'(u), v \right\rangle := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big\{ W(u + \varepsilon v) - W(u) \Big\} \qquad \forall\, v \in V$$

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- SOLVE : $\boxed{\left\langle W'(u), v \right\rangle = 0 \qquad \forall\, v \in V}$ approximately by :
- start with initial guess $u_0 \in V$ and solve **gradient flow eq. :**

Find a function $u : [0, T] \to V$ such that

$$\begin{aligned} (d_t u(t), v)_H &= -\langle W'(u(t)), v \rangle \qquad \forall\, v \in V, \quad t \in [0, T], \\ u(0) &= u_0 \end{aligned}$$

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- weak formulation : Find $u \in u_0 + H^1_{0,-}(I, V)$ , $I := [0, T]$, s. t.

$$\int_0^T (d_t u(t), v(t))_H \, dt = -\int_0^T \left\langle W'(u(t)), v(t) \right\rangle dt \qquad \forall\, v \in L^2(I, V)$$

# Energy decreasing property

- choose test function $v\Big|_{I_n} = d_t u$ and $v\Big|_{I\setminus I_n} = 0_V$, $\quad I_n = [t_{n-1}, t_n]$

$$\int_{I_n} (d_t u(t), d_t u(t))_H \, dt = -\int_{I_n} \left\langle W'(u(t)), d_t u(t) \right\rangle dt$$

$$\Rightarrow \int_{I_n} \|d_t u(t)\|_H^2 dt = -\int_{I_n} d_t \Big\{ W(u(t)) \Big\} dt = W(u(t_{n-1})) - W(u(t_n))$$

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- $\Rightarrow$ **energy decreasing property :**

$W(u(t_n)) < W(u(t_{n-1}))$ as long as $\|d_t u(t)\|_{L^2(I_n,H)} > 0$

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$\Rightarrow$ $u^*$ is candidate for **local minimizer**

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⇒ $u^*$ is candidate for **local minimizer**

- AIM : find **time discretization** with energy decreasing property

# dGP – discontinuous Galerkin-Petrov method

- given data : $u_0 \in V$, $F : [0,T] \times V \to V'$
- continuous problem: Find $u : I \to V$ such that

$$\begin{aligned} d_t u(t) &= F(t, u(t)) \qquad \forall\, t \in I = [0,T], \quad \text{in} \quad V', \\ u(0) &= u_0 \end{aligned}$$

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- weak formulation: Find $u \in u_0 + X_0$ such that

$$\int_0^T \langle d_t u(t), v(t) \rangle \; dt = \int_0^T \langle F(t, u(t)), v(t) \rangle \, dt \quad \forall\, v \in Y = L^2(I, V)$$

where $X_0 := \{u \in L^2(I,V) : \; d_t u \in L^2(I,V'), \; u(0) = 0\}$

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- **discrete solution and test space :**

$$X_\tau^k := \{ u \in C(I,V) : \; u\big|_{I_n} \in \mathbb{P}_k(I_n,V) \quad \forall\, n = 1, \ldots, N \}$$

$$Y_\tau^k := \{ v \in L^2(I,V) : \; v\big|_{I_n} \in \mathbb{P}_{k-1}(I_n,V) \quad \forall\, n = 1, \ldots, N \}$$

# dGP(k) – method

- let $X^k_{\tau,0} := X^k_\tau \cap X_0$
- **dGP(k) - method :** Find $u_\tau \in u_0 + X^k_{\tau,0}$ such that

$$\int_0^T \langle d_t u_\tau(t), v_\tau(t)\rangle \, dt = \int_0^T \langle F(t, u_\tau(t)), v_\tau(t)\rangle \, dt \qquad \forall\, v_\tau \in Y^k_\tau$$

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- **key property :**

$$\boxed{d_t u_\tau \in Y_\tau^k \quad \forall \, u_\tau \in X_\tau^k} \quad \Rightarrow \quad \text{energy decreasing property}$$

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- **time marching process :** let $v_\tau(t) = \psi(t) v$ , $\psi(t) = 0 \ \forall\, t \notin I_n$

$$\int_{I_n} \langle d_t u_\tau(t), v\rangle \, \psi(t) \, dt = \int_{I_n} \langle F(t, u_\tau(t)), v\rangle \, \psi(t) \, dt$$

$$\forall\, v \in V, \quad \psi \in \mathbb{P}_{k-1}(I_n)$$

$\Rightarrow$ $k$ equations for $k$ "unknowns" $U^j_n \in V$ of $u_\tau(t)$ on $I_n$

# Choice of basis functions

- **ansatz :** find $U_n^j \in V$ with $U_n^0 = u_\tau\big|_{I_{n-1}}(t_{n-1})$ such that

$$u(t) \approx u_\tau(t) := \sum_{j=0}^{k} U_n^j \phi_{n,j}(t) \qquad \forall\, t \in I_n,$$

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- choose $\phi_{n,j} \in \mathbb{P}_k(I_n)$ such that

$\boxed{\phi_{n,j}(t_{n,i}) = \delta_{i,j}}$ where $t_{n,j} \in I_n$ **Gauß-Lobatto points**

$$\int_{I_n} \langle d_t u_\tau(t), v\rangle\, \psi_{n,i}(t)\, dt = \sum_{j=0}^{k} \left(U_n^j, v\right)_H \int_{I_n} \phi'_{n,j}(t)\psi_{n,i}(t)\, dt$$

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- choose $\psi_{n,j} \in \mathbb{P}_{k-1}(I_n)$ such that

$$\alpha_{i,j} := \int_{I_n} \phi'_{n,j}(t)\psi_{n,i}(t)\, dt = \delta_{i,j} \qquad \forall i = 1, \ldots, k-1,\ 1 \le j \le k$$

$$\beta_{i,\mu} := w_{n,\mu}\psi_{n,i}(t_{n,\mu}) = 0 \qquad \forall\ \mu = i$$

# system of equations on $I_n$

- $\psi_{n,i}$ are constructed via reference mapping from

$$\hat{\psi}_i(\hat{t}) = \frac{-1}{\hat{w}_i} \int_{\hat{t}_i}^{\hat{t}} \prod_{\substack{\mu=1 \\ \mu\neq i}}^{k-1} \frac{s-\hat{t}_\mu}{\hat{t}_i-\hat{t}_\mu}\, ds \qquad \forall\, i \in \{1,\dots,k-1\}, \quad \hat{\psi}_k(\hat{t}) = 1$$

- system for the $U_n^j$ , $1 \le j \le k$ :

$$\sum_{j=0}^{k} \alpha_{i,j} \left(U_n^j, v\right)_H = \frac{\tau_n}{2} \sum_{j=0}^{k} \beta_{i,j} \left\langle \underbrace{F\left(t_{n,j}, U_n^j\right)}_{F_n^j(U_n^j)}, v \right\rangle \quad \forall\, v \in V,\ 1 \le i \le k$$

reduces to :

$$MU_n^i = -\alpha_{i,0} MU_n^0 - \alpha_{i,k} MU_n^k + \frac{\tau_n}{2} \sum_{\substack{j=0 \\ j\neq i}}^{k} \beta_{i,j} F_n^j(U_n^j),$$

for $1 \le i \le k-1$

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- solve **fixed point equation :**

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- **accuracy :** $\|u - u_\tau\|_{C(I,V)} \le C\tau^3$, even super-conv. $\mathcal{O}(\tau^4)$ !!

# Theoretical properties of the dGP(k) - method

## Theorem (Sch., 2009)

*Assume* $V = H = V' = \mathbb{R}^d$ *and*

- $\|F(t,u^1) - F(t,u^2)\|_V \leq L\|u^1 - u^2\|_V \quad \forall\, u^1, u^2 \in V$
- $L\tau \leq \delta_0$ *sufficiently small*

*Then, dGP(k)-method*

- *a unique solution* $u_\tau$ *exists for the dGP(k)-method*
- *it holds the error estimate*

$$\|u - u_\tau\|_{C(I_n,V)} \leq C \max_{1\leq\mu\leq n} \tau_\mu^{k+1} \|d_t^{k+1} u\|_{C(I_\mu,V)}$$

*where* $C = \tilde{C} e^{2Lt_{n-1}}$*.*

*Furthermore, the dGP(k)-method is* **A-stable***.*

# Simple numerical example

- **problem :** Find $u : [0,1] \to \mathbb{R}^{2\times 2}$ such that

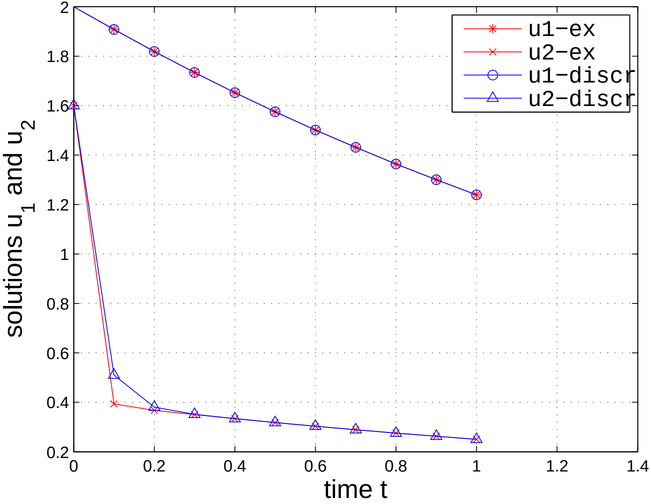
$$\begin{aligned} d_t u(t) &= A u(t) \quad \forall\, t \in [0,1], \\ u(0) &= u_0, \end{aligned}$$

$$A = \begin{pmatrix} -0.5 & 0.1 \\ 10 & -50 \end{pmatrix}, \quad u_0 = \begin{pmatrix} 2 \\ 1.6 \end{pmatrix}$$

- eigenvalues of $A$ : $\lambda_1 \approx -0.48$, $\lambda_2 \approx -50$ $\Rightarrow$ relatively stiff
- we use dGP(2)-method (exact here) with $\tau_n = \tau \ \forall\, n$
- **error norm :**

$$\|u - u_\tau\|_\infty := \max_{1\le n\le N} \|u(t_n) - u_\tau(t_n)\|$$

# Plot of the components of the solution: $\tau = \frac{1}{10}$



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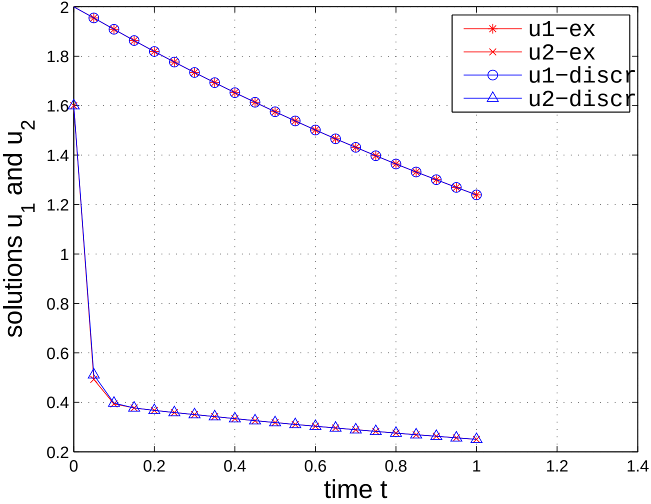

# Error norms

$$\|u - u_\tau\|_\infty := \max_{1 \le n \le N} \|u(t_n) - u_\tau(t_n)\|$$

| $\tau$ | $\|u - u_\tau\|_\infty$ | EOC |
|---|---|---|
| 1/10 | 1.170 e-01 | |
| 1/20 | 1.875 e-02 | 2.6413 |
| 1/40 | 1.592 e-03 | 3.5583 |
| 1/80 | 9.301 e-05 | 4.0968 |
| 1/160 | 5.858 e-06 | 3.9891 |
| 1/320 | 3.645 e-07 | 4.0063 |

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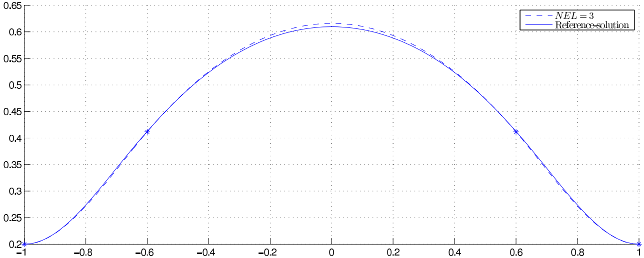
  $V$ = normal velocity of $\Gamma(t)$
- axisymmetric surface $\Gamma(t)$ : $u(x,t)$ = radius at $x \in \Omega = (-1,1)$



- $X_h := \{\phi_h \in C^1(\bar{\Omega}) \,|\, \phi_{|[x_{j-1},x_j]} \in \mathbb{P}_3, 1 \le j \le N\} \subset H^2(\Omega)$.

# Time discretization

**ODE system** for the FE nodal vector: Find $U : [0,T] \to \mathbb{R}^d$ such that

$$\boxed{\begin{array}{rcl} M(U(t))\,U'(t) & = & F(t,U(t)) \qquad \forall\, t \in (0,T] \\ U(0) & = & U^0 \end{array}}$$

with nonlinear mass-matrix

$$M(U) = (M(U)_{i,j}) \qquad \text{with} \qquad M(U)_{i,j} := \int_\Omega \frac{u_h(U)\varphi_j\varphi_i}{\sqrt{1+u_{hx}(U)^2}}$$

where $u_h(U) \in X_h$ denotes

$$u_h(U)(x) = \sum_{j=-1}^{2N} U_j\varphi_j(x), \qquad \forall\, x \in \Omega$$

$$F_i(t,U) := \int_I f(\cdot,t)u_h(U)\varphi_i \;-\; \langle \tilde{W}'(u_h(U)),\varphi_i\rangle$$

# Applying dGP(1) = Crank-Nicolson

- **equivalent ODE system :**

$$\begin{array}{rcl} U'(t) & = & \tilde{F}(t, U(t)) := M(U(t))^{-1} F(t, U(t)) \qquad \forall\, t \in (0, T] \\ U(0) & = & U^0 \end{array}$$

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- **Crank-Nicolson :** solve for $U^n$

$$U^n = U^{n-1} + \frac{\tau}{2} \left\{ \tilde{F}(t_{n-1}, U^{n-1}) + \tilde{F}(t_n, U^n) \right\}$$

by a quasi-Newton defect-correction iteration

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- $L^2$ - error $\sim\ \mathcal{O}(h^4) + \mathcal{O}(\tau^2)$

| $h$ | $\tau$ | $\|e\|_0$ | EOC | $\|e\|_1$ | EOC |
|---|---|---|---|---|---|
| $1/2$ | $1/256$ | $1.245\,e-04$ | | $8.691\,e-04$ | |
| $1/4$ | $1/1024$ | $7.774\,e-06$ | 4.001 | $1.084\,e-04$ | 3.003 |
| $1/8$ | $1/4096$ | $4.860\,e-07$ | 4.000 | $1.355\,e-05$ | 3.001 |
| $1/16$ | $1/16384$ | $3.038\,e-08$ | 4.000 | $1.693\,e-06$ | 3.000 |