Finite Volume schemes based on differential forms - Towards a unified framework for space-time manifolds

Jan Giesselmann

IANS, Universität Stuttgart

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Outline

- Motivation
- Conservation laws given by differential forms
- Entropy solutions and boundary conditions
- Finite Volume schemes
- Prospects

Conservation laws on curved manifolds

Many conservation laws are naturally posed on manifolds, e.g.

- Shallow water equations on the surface of the earth

$$\begin{pmatrix} h \\ hv \end{pmatrix}_t + \nabla_g \cdot \begin{pmatrix} hv \\ hv \otimes v + \frac{h^2}{2g} \end{pmatrix} = 0 \quad \text{in } S^2 \times (0, T),$$

  where $\nabla_g \cdot$ denotes the divergence operator on the sphere. In polar coordinates $\nabla_g \cdot$ is given by

$$\nabla_g \cdot f := \frac{1}{\sin\theta} \left((f^\phi \sin\theta)_\phi + (f^\theta \sin\theta)_\theta \right).$$

- MHD shallow water equations approximating the heliosphere of the sun.

Conservation laws on curved manifolds

- Einstein Euler equations in general relativity,

$$\begin{aligned} R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R &= 8\pi((\epsilon + p)u^\alpha u^\beta + p g^{\alpha\beta}) \\ \nabla_\alpha((\epsilon + p)u^\alpha u^\beta + p g^{\alpha\beta}) &= 0, \\ + \text{ consitutive relations} & \end{aligned}$$

where

R : scalar curvature

$R_{\alpha\beta}$: Ricci curvature tensor

p : pressure

ϵ : energy density

We will consider scalar conservation laws as a simplified model.

Results for special classes of manifolds

- Amorim, Ben-Artzi, LeFloch '05: Convergence of Finite Volume schemes for hyperbolic conservation laws on Riemannian manifolds
- Ben-Artzi, LeFloch '07: Well-posedness theory for hyperbolic conservation laws on Riemannian manifolds
- Amorim, LeFloch, Okutmustur '08: Convergence of Finite Volume schemes for hyperbolic conservation laws on Lorentzian manifolds
- Giesselmann '09, LeFloch, Neves, Okutmustur '09: Convergence rate for Finite Volume schemes for hyperbolic conservation laws on Riemannian manifolds

Aim: generalised framework for general manifolds without using a Riemannian or Lorentzian structure.

Conservation laws on general manifolds

- A **flux field** on a smooth, oriented $(n+1)$-dimensional manifold M is given by a parametrised family $\omega(\bar{u})$ of smooth n-forms that depends smoothly upon the real parameter $\bar{u}$.
- The conservation law associated with the flux field ω and with unknown $u : M \to \mathbb{R}$ reads

  $$d(\omega(u)) = 0, \quad \text{(CONS)}$$

  where d denotes the exterior differential and hence $d(\omega(u))$ is a $(n+1)$-form on M.

Conservation laws on Riemannian manifolds

Let (N,g) be a Riemannian manifold, then a conservation law on $[0,\infty)\times N$ reads

$$u_t + \nabla_g \cdot f(x,u) = 0,$$

where $f(x,u)$ is the flux vector field. In local coordinates this takes the form

$$u_t + \sum_{i=1}^{n} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(f^i(x,u)\sqrt{|g|(x)} \right) = 0,$$

where f has the representation $f = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i}$ in local coordinates and $|g|$ denotes the determinant of the metric tensor. This can be stated in the form (CONS) by defining

$$\omega(x,u) := \sqrt{|g|}u\, dx^1\wedge\cdots\wedge dx^n + \sum_{i=1}^{n}(-1)^i\sqrt{|g|}f^i(x,u)\, dx^0\wedge\ldots\widehat{dx^i}\cdots\wedge dx^n,$$

where we identify $x^0 = t$.

Conservation laws on Lorentzian manifolds

Let (M, g) be a (n+1)-dimensional Lorentzian manifold, i.e. g is a semi-Riemannian metric with signature $(-,+,+,+)$ then a conservation law on M reads

$$\nabla_g \cdot (f(x,u)) = 0,$$

where $f(x,u)$ is the flux vector field. In local coordinates this takes the form

$$\frac{\partial}{\partial x^i}\left(f^i\sqrt{|g|}\right) = 0,$$

where f has the representation $f = \sum_{i=0}^n f^i \frac{\partial}{\partial x^i}$ in local coordinates and $|g|$ denotes the determinant of the metric tensor. This can be stated in the form (CONS) by defining

$$\omega(x,u) := \sum_{i=0}^{n} (-1)^i \sqrt{|g|} f^i(x,u)\, dx^0 \wedge \dots \hat{dx^i} \cdots \wedge dx^n.$$

Global hyperbolicity

A conservation law in Euclidean space

$$g(u)_{x_1} + f(u)_{x_2} = 0$$

is called hypercolic if there exist $a, b \in \mathbb{R}$ such that

$$ag'(u) + bf'(u) \neq 0 \ \forall u \in \mathbb{R}.$$

The flux field ω of the conservation law (CONS) satisfies the **global hyperbolicity condition** if there exists a smooth global 1-form T such that

$$T \wedge \partial_u \omega(\bar{u}) > 0 \ \forall \bar{u} \in \mathbb{R},$$

where $'' >''$ means that it is positive on oriented bases of tangent spaces.
We call T a field of **observers**.
For Riemannian manifolds the differential dt obviously yields an observer because

$$dt \wedge \partial_u \omega(\bar{u}) = \sqrt{|g|} dt \wedge dx^1 \wedge \cdots \wedge dx^n > 0.$$

Boundary conditions & Geometry compatibility

We restrict our attention to a special class of flux fields, ensuring that constant functions are solutions
A flux field ω is called **geometry compatible** if

$$(d\omega)(\bar{u}) = 0 \quad \forall \bar{u} \in \mathbb{R}.$$

Let $u_B : \partial M \to \mathbb{R}$. We will consider the boundary value problem

$$\begin{aligned} d(\omega(u)) &= 0 \text{ on } M, \\ u &= u_B \text{ on } \partial M. \end{aligned}$$

Like in the Euclidean case this boundary condition has to be understood in a sufficiently weak sense. It is important to note that we cannot exactly distinguish the initial condition from other boundary conditions.

Entropy pairs in the Euclidean setting

$u : D \subset \mathbb{R}^n \to \mathbb{R}$ is called an entropy solution of the boundary value problem

$$u_t + \operatorname{div} f(u) = 0 \quad u(\cdot,0) = u_0 \text{ on } D, \quad u(\cdot,t) = u_B(\cdot,t) \text{ on } \partial D$$

if there exists a function $b : \partial D \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\begin{aligned}\int_{D\times\mathbb{R}_+} U(u)\partial_t\psi + F(u)\text{grad } \psi \, dxdt + \int_D U(u(0))\psi(x,0)dx \\ - \int_{\partial D\times\mathbb{R}_+} \left(F(u_B)\cdot\nu + U'(u_B)\left(b - f(u_B)\cdot\nu\right)\right)\psi \, d\Gamma dt \qquad \geq 0,\end{aligned}$$

where ν is the outward unit normal to ∂D for every convex entropy pair (U,F), i.e. $F' = U'f'$, and test function $\psi \in C_0^\infty(D \times \mathbb{R}_+)$.

Entropy pairs

A parametrised family of smooth n-forms $\Omega(\bar{u})$ depending Lipschitz continuously on $\bar{u} \in \mathbb{R}$ is called a (convex) entropy flux field for the conservation law (CONS) when there exists a (convex) function $U : \mathbb{R} \to \mathbb{R}$ such that

$$\Omega(\bar{u}) = \int_0^{\bar{u}} \partial_u U(\bar{v}) \partial_u \omega(\bar{v}) \, d\bar{v} \quad \forall \bar{u} \in \mathbb{R}.$$

The set (U, Ω) is called an **entropy pair**. Then every smooth solution u of (CONS) satisfies

$$d(\Omega(u)) - (d\Omega)(u) + \partial_u U(u)(d\omega)(u) = 0.$$

Entropy pairs

Now we can generalise the notion of entropy solutions to general manifolds: A function $u : M \to \mathbb{R}$ is called an **entropy solutions** of the boundary value problem if there exists a smooth field of n-forms on the boundary ∂M such that

$$\int_M (d\psi \wedge \Omega(u)) + \int_{\partial M} \psi \left(i^*\Omega(u_B) + \partial_u U(u_B)(\gamma - i^*\omega(u_B))\right) \geq 0,$$

where $i^*\omega$ is the pullback of ω along the inclusion $i : \partial M \to M$, for every convex entropy pair (U, Ω) and test function $\psi \in C_0^\infty(M)$.

Spacelike hypersurfaces

If we look at a space-time cell $C \times [t_n, t_{n+1}]$ in the Euclidean or Riemannian setting we have two distinguished faces namely

$$C \times \{t_n\} \text{ and } C \times \{t_{n+1}\}$$

and it is straightforward to calculate the value of the approximate solution u^h when we know the flux through $C \times \{t_{n+1}\}$. Equivalently we can say

$$\bar{u} \rightarrow \int_{C \times \{t_{n+1}\}} \bar{u}$$

is bijective. This situation is more involved in the case of general manifolds.

Spacelike hypersurfaces

A hypersurface $\mathcal{H}$ of M is called **spacelike** if for every normal 1-form N

$$N \wedge \partial_u \omega(\bar{u}) \neq 0, \quad \bar{u} \in \mathbb{R}.$$

This implies that the function

$$\mathbb{R} \to \mathbb{R}, \quad \bar{u} \mapsto \int_{\mathcal{H}} i^* \omega(\bar{u})$$

satisfies

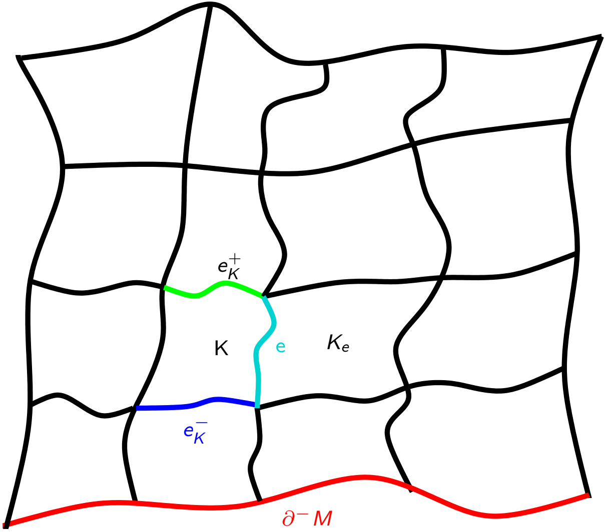
$$\partial_u \int_{\mathcal{H}} i^* \omega(\bar{u}) > 0.$$

We need a somewhat sharper condition: For every spacelike hypersurface $\mathcal{H}$ there exists some $c_{\mathcal{H}} > 0$ such that

$$\partial_u \int_{\mathcal{H}} i^* \omega(\bar{u}) > c_{\mathcal{H}}$$

such that the map is bijective.

The grid



Orientations

For every cell K there have to be at least two space-like faces. One e_K^- playing the role of $C \times \{t_n\}$ and one e_K^+ playing the role of $C \times \{t_{n+1}\}$.

The set of "other" faces of K is denotes by $\partial^0 K$.

The orientations of the faces e of C are chosen in such a way that Stokes theorem implies

$$\int_{e_K^+} i^*\omega(u) = \int_{e_K^-} i^*\omega(u) - \sum_{e\in\partial^0 K} \int_e i^*\omega(u),$$

for the exact solution u and every cell $K \in \mathcal{T}$. In the Euclidean case this reads

$$\int_C u(t_{n+1})\, dx = \int_C u(t_n)\, dx - \sum_{e\in\partial C} \int_{t_n}^{t_{n+1}} \int_C f(u(t)) \cdot \nu \, d\Gamma dt.$$

Numerical fluxes

To approximate the sum

$$\sum_{e\in\partial^0 K}\int_e i^*\omega(u)$$

we introduce for every $K\in\mathcal{T}$ and $e\in\partial^0 K$ a locally Lipschitz continuous numerical flux function

$$Q_{K,e}:\mathbb{R}^2\to\mathbb{R}$$

satisfying

$$\begin{aligned}
Q_{K,e}(\bar u,\bar u)&=\int_e i^*\omega(\bar u) &&:\quad \text{Consistency,}\\
Q_{K,e}(\bar u,\bar v)&=-Q_{K_e,e}(\bar v,\bar u) &&:\quad \text{Conservation,}\\
\frac{\partial}{\partial\bar u}Q_{K,e}(\bar u,\bar v)\geq 0,\ \frac{\partial}{\partial\bar v}Q_{K,e}(\bar u,\bar v)&\leq 0 &&:\quad \text{Monotonicity.}
\end{aligned}$$

The iterative scheme

For cells K with $e_K^- \subset \partial^- M$ we define $u_{e_K^-}$ by imposing

$$\int_{e_K^-} i^*\omega(u_{e_K^-}) = \int_{e_K^-} i^*\omega(u_B).$$

Then we define for every cell K the value $u_{e_K^+}$ by

$$\int_{e_K^+} i^*\omega(u_{e_K^+}) = \int_{e_K^-} i^*\omega(u_{e_K^-}) - \sum_{e\in\partial^0 K} Q_{K,e}(u_{e_K^-}, u_{e_{K_e}^-}).$$

This is well-defined because e_K^+ is spacelike. When e is part of the boundary we fix a non vanishing n-form α_e on e and define

$$u_{e_{K_e}^-} := \frac{\int_e u_B\alpha_e}{\int_e \alpha_e}$$

which is independent of the orientation of e.

Boundedness of solutions

Proposition: Provided the CFL condition

$$\sum_{e \in \partial^0 K} \sup_{u,v} \left| \frac{\partial_u Q_{K,e}(u,v) - \partial_v Q_{K,e}(u,v)}{\partial_u \int_{e_K^+} i^* \omega(u)} \right| \leq \frac{1}{2}$$

for each cell K the approximate solution given by the finite volume scheme satisfies

$$u_{e_K^+} \in [\operatorname{essinf} u_B, \operatorname{esssup} u_B] \quad \forall K \in \mathcal{T}.$$

Remark: The only structure on the manifold needed to define the finite volume scheme is given by the flux field ω.

Convergence

Nevertheless if we want to prove convergence, we have to introduce a metric d on M and we need a constant $c > 0$ such that we have a sequence $\{h_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_+$ with $h_i \to 0$ and grids $\mathcal{T}^{h_i}$ satisfying

$$\begin{aligned} d(x,y) &\leq ch_i \quad \forall \quad x,y \in K : K \in \mathcal{T}^{h_i}, \\ \sup_{\bar{u}} \partial_u \int_{e_K^+} i^*\omega(\bar{u}) &\leq ch_i^n \quad \forall \quad K \in \mathcal{T}^{h_i}, \end{aligned}$$

plus some technical conditions and a CFL condition.

Convergence in Euclidean space

One of the main steps in the convergence proof in the Euclidean setting is the following

Lemma (Kröner & Rokyta '94, Eymard et.al. '98):

$$
\begin{aligned}
&-\sum_n \sum_{C\in\mathcal{T}^h} \int_{t_n}^{t_{n+1}} \int_C U(u_C^n)\psi_t + F(u_C^n)\nabla\psi \, dx\, dt + \sum_{C\in\mathcal{T}^h} u_C^0 \psi \, dx \\
&+\sum_n \sum_{e\in\partial\mathcal{T}^h} \int_{t_n}^{t_{n+1}} \int_e F(u_{C_e}^n)\nu_{e,C} \, d\Gamma dt \\
&+\sum_n \sum_{e\in\partial\mathcal{T}^h} \int_{t_n}^{t_{n+1}} \int_e U'(u_C^n)\left(Q_{C,e}(u_C^n, u_{C_e}^n) - f(u_{C_e}^n)\nu_{e,C}\right)\psi \, d\Gamma dt \\
&\leq R(h),
\end{aligned}
$$

where $R(h) \to 0$ for $h \to 0$ and $\mathcal{T}^h$ is here a triangulation of N.

Convergence

Lemma (Giesselmann, LeFloch '09):

$$- \sum_{K\in\mathcal{T}^{h_i}} \int_K d\psi \wedge \Omega(u_{e_K^-}) + \sum_{K\in\mathcal{T}_-^{h_i}} \int_{e_K^-} \psi i^*\Omega(u_{e_K^-}) + \sum_{e^0\in\mathcal{T}_0^{h_i}} \int_{e^0} \psi_{e^0} i^*\Omega(u_{e_{K_{e^0}}^-})$$

$$+ \sum_{e^0\in\mathcal{T}_0^{h_i}} \psi_{e^0}\partial_u U(u_{e_K^-}) \left(Q_{K,e^0}(u_{e_K^-}, u_{e_{K_{e^0}}^-}) - Q_{K,e^0}(u_{e_{K_{e^0}}^-}, u_{e_{K_{e^0}}^-}) \right) \le R(h)$$

where

$$\begin{array}{rcl} R(h) \to 0 & \text{for} & h \to 0, \\ \mathcal{T}_-^{h_i} & := & \{K \in \mathcal{T}^{h_i} : e_K^- \subset \partial^- M\}, \\ \mathcal{T}_0^{h_i} & := & \{e \in \partial^0 K : K \in \mathcal{T}^{h_i}, e \subset \partial M\}, \\ \psi_{e^0} & := & \dfrac{\int_{e^0} u_B \alpha_{e^0}}{\int_{e^0} \alpha_{e^0}}. \end{array}$$

Prospects

- Convergence of the finite volume scheme,
- Existence of solutions,
- L^1 contraction property.