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# Higher-order FEM and pointwise error estimates for elliptic problems on surfaces

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1. Higher-order FEM are often more efficient.
2. Higher-order surface FEM may be more convenient for other reasons (preservations of curvature information...).
3. Surface FEM are widely used, so more extensive error analysis is worthwhile.
4. Pointwise error estimates are of importance in nonlinear problems.

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## Literature:

- [De. '09, SINUM]

# Preliminaries

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## Definitions:

- $\Gamma$  is a compact, 2-dimensional  $C^2$  surface without boundary in  $\mathbb{R}^3$ .
- $d\sigma$  is surface measure on  $\Gamma$ .
- $f$  is (given) data satisfying  $\int_{\Gamma} f d\sigma = 0$ .
- $\nabla_{\Gamma}$  is the tangential gradient.
- $\Delta_{\Gamma} = \nabla_{\Gamma} \cdot \nabla_{\Gamma}$  is the Laplace-Beltrami operator on  $\Gamma$ .
- $H^1(\Gamma) = \{u \in L_2(\Gamma) : \nabla_{\Gamma} u \in L_2(\Gamma)\}$ .

## Laplace-Beltrami problem (weak form):

$$\int_{\Gamma} \nabla_{\Gamma} u \nabla_{\Gamma} v d\sigma = \int_{\Gamma} f v d\sigma \text{ for all } v \in H^1(\Gamma).$$

We require  $\int_{\Gamma} u d\sigma = 0$  to ensure uniqueness.

# Dziuk's surface FEM

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- *Discrete surface*:  $\Gamma_h$  is a polyhedron with shape-regular triangular faces of diameter  $h$ , vertices on  $\Gamma$ .
- *Finite element space*:  $S_h$  is the piecewise linear functions on  $\Gamma_h$ .
- *Data*:  $f$  is defined on  $\Gamma$ , so have to define  $f_h$  on  $\Gamma_h$ .
- *Finite element method*: Find  $\tilde{u}_h \in S_h$  such that  $\int_{\Gamma_h} \tilde{u}_h d\sigma_h = 0$  and

$$\int_{\Gamma_h} \nabla_{\Gamma_h} \tilde{u}_h \nabla_{\Gamma_h} v_h d\sigma_h = \int_{\Gamma_h} f_h v_h d\sigma_h, \quad v_h \in S_h.$$

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## Two sources of numerical error:

1. *PDE/Galerkin error* results from replacing an infinite-dimensional space with a finite-dimensional space;
2. *Geometric error* results from approximating  $\Gamma$  by  $\Gamma_h$  (*variational crime*).

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**Goal:** Analyze error in terms of geometric quantities.

# Implicit representation of surfaces

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**Note:** Assumption is true if  $\Gamma$  is  $C^2$  (cf. Gilbarg & Trudinger).

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**Then:** For  $x$  lying in a sufficiently small strip  $U$  about  $\Gamma$ ,

- *Distance function:*  $|d(x)| = \text{dist}(x, \Gamma)$ .
- *Normal vector:*  $\vec{\nu} = \nabla d$  satisfies  $|\vec{\nu}| = 1$ ;  $\vec{\nu}$  is a unit normal on  $\Gamma$ .
- *Weingarten map:*  $\mathbf{H} = \nabla^2 d = \nabla \vec{\nu}$ . Non-zero eigenvalues  $\kappa_1, \kappa_2$  are principal curvatures.
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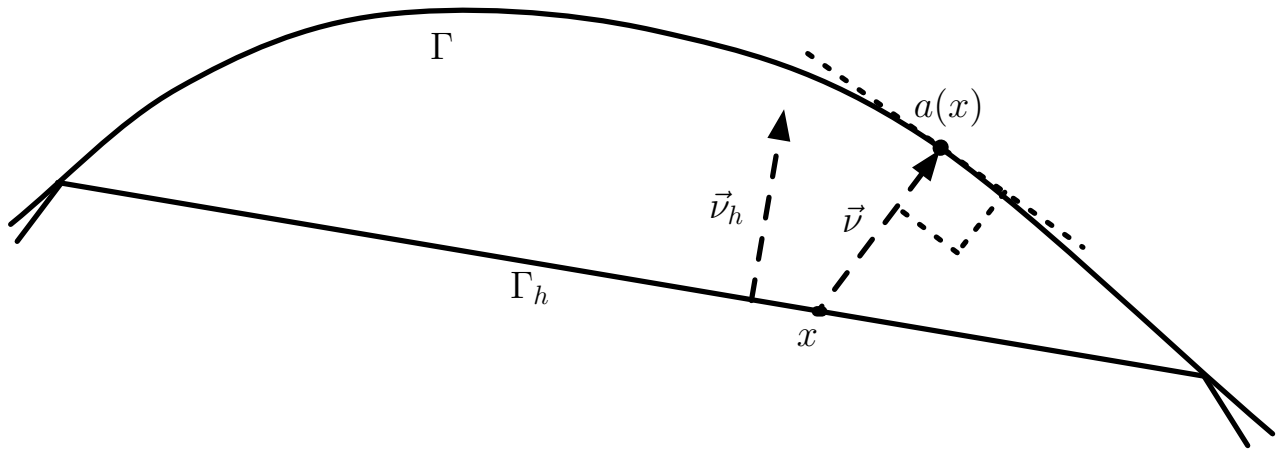
**But, for practical computations:**

- In most cases,  $d$  is not explicitly available. Exceptions: Sphere, torus.
- Usually,  $\Gamma$  is the zero level set of  $\zeta$  with  $\zeta$  NOT a distance function.
- $a$  must be approximated numerically; can extract  $\nu$  and  $\mathbf{H}$  if needed (cf. [De.-Dziuk '07]).

# Relationship between $\Gamma$ and $\Gamma_h$

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Basic assumption on  $\Gamma_h$ :  $\Gamma_h \subset U$  (so  $a$  is a bijection between  $\Gamma$  and  $\Gamma_h$ ).



## $\Gamma$ and $\Gamma_h$ , cont.

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Definitions for comparing  $H^1$ -inner products on  $\Gamma$  and  $\Gamma_h$ :

- For  $x \in \Gamma_h$ , let  $\mu_h(x) := \frac{d\sigma(a(x))}{d\sigma_h(x)}$ , where  $d\sigma$  is surface measure on  $\Gamma$ .
- *Lifts of functions:* For  $v(x)$  defined on  $\Gamma_h$ ,  $v^\ell(a(x)) := v(x)$  is defined on  $\Gamma$ .
- *Projections:*  $\mathbf{P}$  and  $\mathbf{P}_h$  are projections onto the tangent planes of  $\Gamma$  and  $\Gamma_h$ , respectively.
- *Dirichlet form identity:* With  $\mathbf{A}_h := \frac{1}{\mu_h} \mathbf{P}(\mathbf{I} - d\mathbf{H})\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P}$ ,

$$\begin{aligned} \int_{\Gamma_h} \nabla_{\Gamma_h} v_h \nabla_{\Gamma_h} \psi_h \, d\sigma_h &= \int_{\Gamma} \mathbf{A}_h \nabla_{\Gamma} v_h^\ell \nabla_{\Gamma} \psi_h^\ell \, d\sigma \\ &= \int_{\Gamma} \nabla_{\Gamma} v_h^\ell \nabla_{\Gamma} \psi_h^\ell \, d\sigma + \int_{\Gamma} (\mathbf{A}_h - \mathbf{P}) \nabla_{\Gamma} v_h^\ell \nabla_{\Gamma} \psi_h^\ell \, d\sigma. \end{aligned}$$

Note: On a triangle of size  $h$ ,  $\|\mathbf{P} - \mathbf{A}_h\|_{\ell_2} \leq Ch^2$ .

# A priori estimates for Dziuk's method

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**Theorem 1.** (Dziuk88) *For discrete data  $f_h(x) = \mu_h(x)f(a(x))$ ,*

$$\begin{aligned}\|\nabla_{\Gamma}(u - \tilde{u}_h^{\ell})\|_{L_2(\Gamma)} &\leq C \inf_{\chi \in S_h} \|\nabla_{\Gamma}(u - \chi^{\ell})\|_{L_2(\Gamma)} + C \|\mathbf{P} - \mathbf{A}_h\|_{L_{\infty}(\Gamma)} \|\nabla_{\Gamma}u\|_{L_2(\Gamma)} \\ &\leq Ch \|u\|_{H^2(\Gamma)} + Ch^2 \|u\|_{H^1(\Gamma)}\end{aligned}$$

*and*

$$\|u - \tilde{u}_h^{\ell}\|_{L_2(\Gamma)} \leq Ch^2 \|u\|_{H^2(\Gamma)}.$$

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Notes:

- Error contains a PDE part and a geometric part.

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and

$$\|u - \tilde{u}_h^{\ell}\|_{L_2(\Gamma)} \leq Ch^2\|u\|_{H^2(\Gamma)}.$$

Notes:

- Error contains a **PDE part** and a **geometric part**.
- Proof employs a *perturbed Galerkin orthogonality relationship*:

$$\int_{\Gamma} \nabla_{\Gamma}(u - u_h^{\ell}) \nabla_{\Gamma} \chi^{\ell} \, d\sigma = \int_{\Gamma} (\mathbf{P} - \mathbf{A}_h) \nabla_{\Gamma} u_h^{\ell} \nabla_{\Gamma} \chi_h^{\ell} \, d\sigma.$$

# General error analysis

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## Two steps in general error analysis:

1. Carry out analysis assuming abstract approximation properties of FE space on  $\Gamma$  (NOT  $\Gamma_h$ ). Because:
  - Geometric error is conceptually handled separately.
  - “Hard” (non-energy) estimates are portable to other methods of handling surface FEM.
2. Analyze geometric error.

# Assumptions on the finite element space

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Let  $S_h^r \subset H^1(\Gamma)$  have the following properties:

- *Approximation:* There exists an interpolant  $I_h : H_2^2(\Gamma) \rightarrow S_h^r$  such that for  $1 \leq p \leq \infty$ ,  $i = 0, 1$ ,  $2 \leq m \leq r + 1$ , and  $v$  sufficiently smooth,

$$|v - Iv|_{W_p^i} \leq Ch^{m-i} \|v\|_{W_p^m}.$$

- *Superapproximation.* If  $\omega \in W_\infty^{r+1}(\Gamma)$  with  $|\omega|_{W_\infty^i} \leq Cd^{-i}$  for some  $d \geq h$ , and  $\chi \in S_h^r$ , then

$$\|\nabla_\Gamma(\omega\chi - I(\omega\chi))\|_{L_2} \leq C \frac{h}{d^2} \|\chi\|_{L_2} + C \frac{h}{d} \|\nabla_\Gamma\chi\|_{L_2}.$$

- *Inverse estimate.* For any  $\chi \in S_h^r$ ,

$$\|\nabla_\Gamma\chi\|_{L_2} \leq Ch^{-1} \|\chi\|_{L_2}.$$

# Geometric error functional

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**Recall:** In Dziuk's method,

$$\int_{\Gamma} \nabla_{\Gamma}(u - \tilde{u}_h^{\ell}) \nabla_{\Gamma} \chi^{\ell} \, d\sigma = \int_{\Gamma} (\mathbf{P} - \mathbf{A}_h) \nabla_{\Gamma} \tilde{u}_h^{\ell} \nabla_{\Gamma} \chi_h^{\ell} \, d\sigma.$$

**Perturbed Galerkin orthogonality assumption:**  $u_h \in S_h^r$  satisfies

$$\int_{\Gamma} \nabla_{\Gamma}(u - u_h) \nabla_{\Gamma} \chi \, d\sigma = F(\chi) \text{ for all } \chi \in S_h^r,$$

where  $F$  is a continuous linear functional on  $H^1(\Gamma) \setminus \mathbb{R}$ .

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**Can analyze various FEM:**

- “Compute, then lift:”  $u_h$  is computed on  $\Gamma_h$  by Dziuk's method. Then  $F(\chi) = \int_{\Gamma} (\mathbf{P} - \mathbf{A}_h) \nabla_{\Gamma} \tilde{u}_h^{\ell} \nabla_{\Gamma} \chi_h^{\ell} \, d\sigma$ .
- “Lift, then compute:”  $u_h$  is computed directly on  $\Gamma$ . Then  $F(\chi) = 0$  IF  $a$  and quadrature are exact.
- “General FEM:”  $u_h$  is computed on  $\Gamma_h$ , but a *non-orthogonal* bijection between  $\Gamma_h$  and  $\Gamma$  is used instead of  $a$ .  $F \neq 0$  encodes the geometric error.  
*Example:*  $\Gamma$  is a graph; bijection is the obvious “vertical” map.

# Abstract error estimates

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**Theorem 2.** (De.) *Let*

$$\begin{aligned} |||F|||_{H^{-j}} &= \sup_{u \in H^j(\Gamma) \setminus \mathbb{R}, \|v\|_{H^j(\Gamma) \setminus \mathbb{R}}=1} F(v), \\ |||F|||_{W_\infty^{-j}} &= \sup_{u \in W_1^j(\Gamma) \setminus \mathbb{R}, \|v\|_{W_1^j(\Gamma) \setminus \mathbb{R}}=1} F(v). \end{aligned}$$

*If  $u$  and  $\Gamma$  are sufficiently smooth and  $\int_\Gamma (u - u_h) \, d\sigma = 0$ , then*

$$\begin{aligned} \|\nabla_\Gamma(u - u_h)\|_{L_2(\Gamma)} &\leq Ch^r \|u\|_{H^{r+1}(\Gamma)} + C |||F|||_{H^{-1}}, \\ \|u - u_h\|_{L_2(\Gamma)} &\leq Ch^{r+1} \|u\|_{H^{r+1}(\Gamma)} + C(h |||F|||_{H^{-1}} + |||F|||_{H^{-2}}). \end{aligned}$$

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*Let also  $\ell_h \ln \frac{1}{h}$ ,  $\ell_{hr} = \ell_h$  if  $r = 1$ , and  $\ell_{hr} = 1$  if  $r \geq 2$ . Then*

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**Note:** Error splits into a **PDE part** and a “geometric” part.

# Higher-order FEM on surfaces

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**Note:** To achieve an overall higher order of convergence, we have to raise the degree of both the FE space and of  $\Gamma_h$ . Define:

- $a_k : \Gamma_h \rightarrow \mathbb{R}^3$  is the  $k - th$  degree Lagrange interpolant of  $a$  over  $\Gamma_h$ .  
( $k = 2$  gives a quadratic approximation of  $\Gamma$ , etc.)
- $\Gamma_h^k = a_k(\Gamma_h)$ . (Thus  $\Gamma_h = \Gamma_h^1$ .)
- $S_r^k = \{\chi \in C_0(\Gamma_h^k) : \chi(a_k)|_{a_k(T)} \in \mathbb{P}_r(T), T \text{ a face of } \Gamma_h\}$ .
- Finite element approximation: Find  $\tilde{u}_h \in S_r^k$  with  $\int_{\Gamma_h^k} \tilde{u}_h d\sigma_h^k = 0$ ,

$$\int_{\Gamma_h^k} \nabla_{\Gamma_h^k} \tilde{u}_h \nabla_{\Gamma_h^k} \chi d\sigma_h^k = \int_{\Gamma_h^k} \chi f_h d\sigma_h^k \text{ for all } \chi \in S_r^k.$$

Here  $d\sigma_h^k$  is surface measure on  $\Gamma_h^k$ , and  $f_h = \mu_h^k(x) f(a(x))$ .

- Definitions largely taken from [Heine].

# Error estimates for higher-order surface FEM

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**Corollary 3.** (*De.*)

$$\begin{aligned}\|\nabla_{\Gamma}(u - \tilde{u}_h^{\ell})\|_{L_2(\Gamma)} &\leq Ch^r \|u\|_{H^{r+1}(\Gamma)} + Ch^{k+1} \|\nabla_{\Gamma}u\|_{L_2(\Gamma)}, \\ \|u - \tilde{u}_h^{\ell}\|_{L_2(\Gamma)} &\leq Ch^{r+1} \|u\|_{H^{r+1}(\Gamma)} + Ch^{k+1} \|\nabla_{\Gamma}u\|_{L_2(\Gamma)}.\end{aligned}$$

With  $\ell_h$  and  $\ell_{hr}$  as before,

$$\begin{aligned}\|\nabla_{\Gamma}(u - \tilde{u}_h^{\ell})\|_{L_{\infty}(\Gamma)} &\leq Ch^r \|u\|_{W_{\infty}^{r+1}(\Gamma)} + Ch^{k+1} \ell_h \|\nabla_{\Gamma}u\|_{L_{\infty}(\Gamma)}, \\ \|u - \tilde{u}_h^{\ell}\|_{L_{\infty}(\Gamma)} &\leq \ell_{hr} (Ch^{r+1} \|u\|_{W_{\infty}^{r+1}(\Gamma)} + Ch^{k+1} \|\nabla_{\Gamma}u\|_{L_{\infty}(\Gamma)}).\end{aligned}$$

**Notes:**

- Corollary is proved by bounding negative norms of  $F$ .
- To achieve  $r$ -th order convergence in a  $W_p^1$  norm, use an  $r - 1$ -st degree approximation to  $\Gamma$ .
- To achieve  $(r + 1)$ -st order convergence in an  $L_p$  norm, use an  $r$ -th degree approximation to  $\Gamma$ .
- Numerical tests confirm that these estimates are sharp.

# Summary for higher-order FEM on surfaces

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Order of convergence of  $\|u - \tilde{u}_h^\ell\|_{L_p}$ :

		k (surface degree)			
		1 (linear)	2 (quads)	3 (cubic)	4 (quartic)
r	1	2	2	2	2
	2	2	3	3	3
	3	2	3	4	4
	4	2	3	4	5

Order of convergence of  $\|\nabla_\Gamma(u - \tilde{u}_h^\ell)\|_{L_p}$ :

		k (surface degree)			
		1 (linear)	2 (quads)	3 (cubic)	4 (quartic)
r	1	1	1	1	1
	2	2	2	2	2
	3	2	3	3	3
	4	2	3	4	4

# Numerical tests: Linear surface

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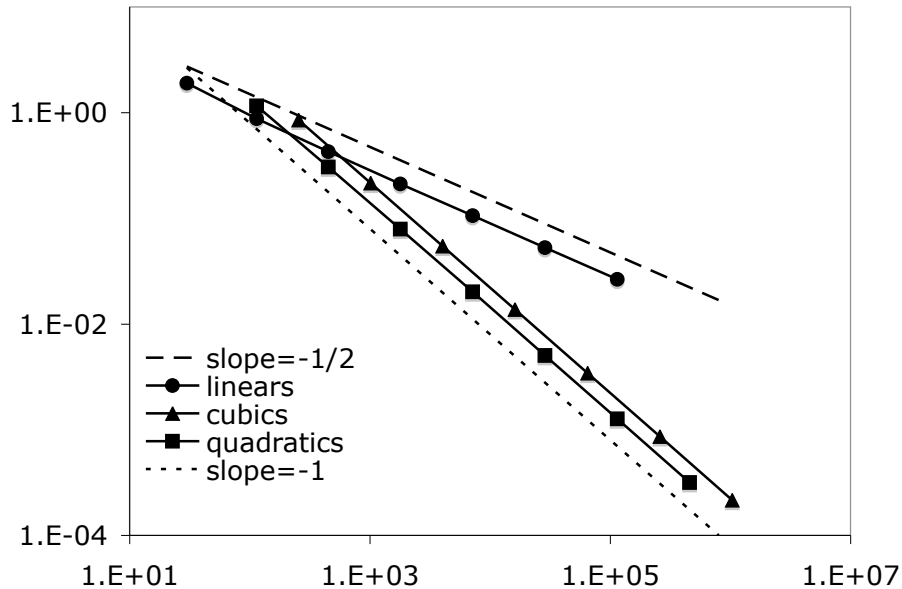


Figure 1: Plot of  $\|\nabla_{\Gamma}(u - u_h)\|_{L_2(\Gamma)}$  vs. the number of degrees of freedom: FEM defined on  $\Gamma_h$ . (Computations: ALBERTA.)

# Numerical tests: Quadratic surface

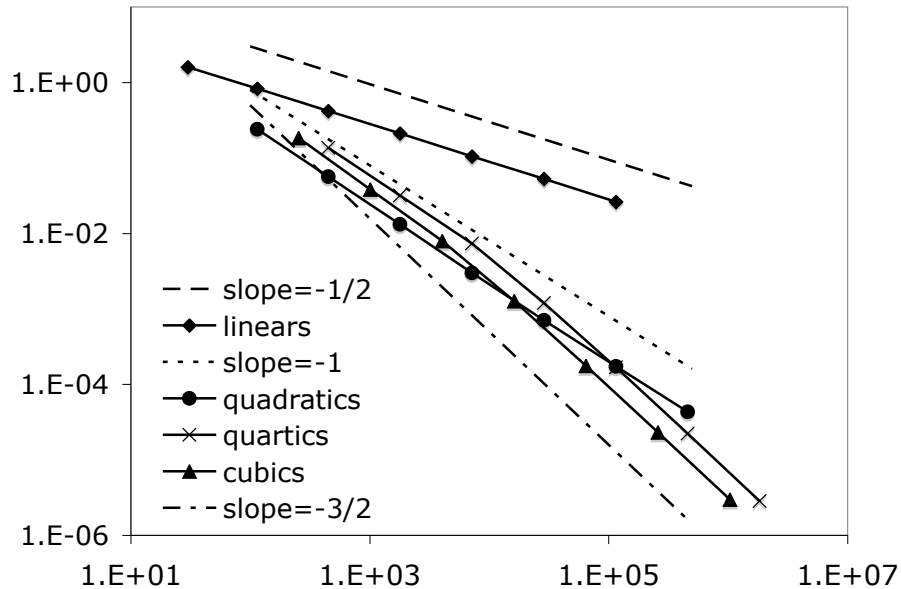


Figure 2: Plot of  $\|\nabla_{\Gamma}(u - u_h)\|_{L_2(\Gamma)}$  vs. the number of degrees of freedom: FEM defined on  $\Gamma_h^2$ .

# Numerical tests on $\Gamma$

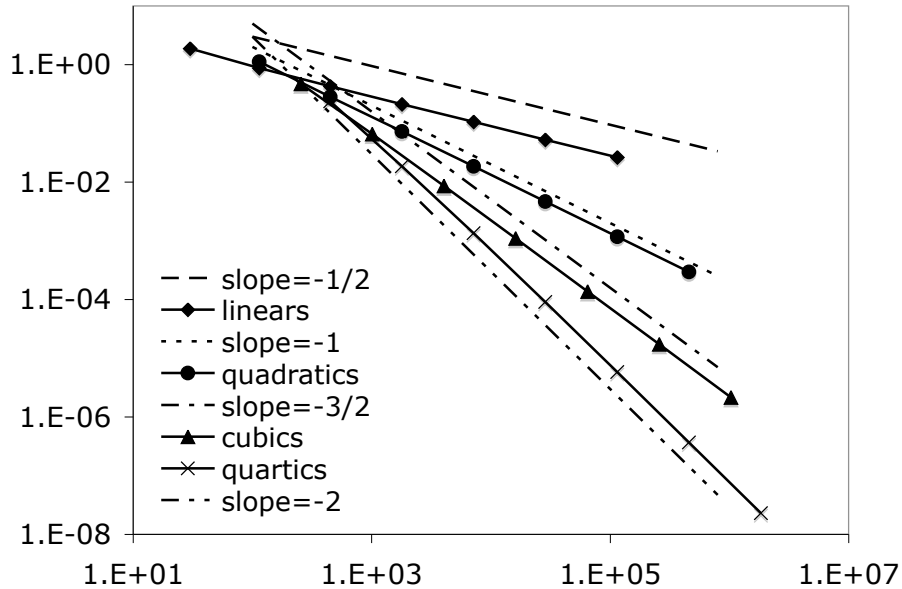


Figure 3: Plot of  $\|\nabla_{\Gamma}(u - u_h)\|_{L_2(\Gamma)}$  vs. the number of degrees of freedom: FEM defined on  $\Gamma$ .

# Extensions

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1. Surfaces with boundary:

- No variational crimes if  $\partial\Gamma$  is “curvi-polygonal” ( $a(\partial\Gamma_h) = \partial\Gamma$ ).
- In contrast to  $\mathbb{R}^n$ , curvi-polygonal domains with smooth boundary exist. Ex.: Half-sphere.

2. Same orders of convergence hold if nodes of  $\Gamma_h^k$  are within  $O(h^{k+1})$  of  $\Gamma$ .

- Likely scenario in evolution problems.

3. General-second order elliptic PDE:

- Energy estimates hold if coefficients are compatible in a natural way.
- Pointwise estimates require Green’s function estimates.

4. Higher-order time discretizations?

5. Other types of discretization? (DG, nonconforming,  $p$ ,  $hp$ ...)